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IDEALS IN LINEAR ALGEBRAS*

BY C. C. MACDUFFEE

1. *Introduction.* In an algebraic field, and indeed in every commutative domain of integrity, the problem of the greatest common divisor and the problem of unique factorization into primes are equivalent. That is, the property that every pair of numbers in the domain of integrity have a greatest common divisor unique up to a unit factor implies the property that factorization be unique save for unit factors and order of multiplication. In those commutative domains of integrity in which these properties do not hold, the introduction of Dedekind ideals gives an extended system in which they do hold.

In a domain of integrity of a non-commutative algebra, the existence of the greatest common right divisor and of the greatest common left divisor does not insure unique factorization. The two problems have diverged and it seems very improbable to the writer that the same type of ideal system will handle both problems.

The Dedekind ideal is by its very definition connected with the problem of the greatest common divisor. This theory, including the theory of the class number, seems to extend practically intact to the non-commutative case. From this standpoint the theory of ideals in linear algebras is successful.

Several writers have attempted to apply Dedekind ideals to the problems of unique factorization, and in every case the result has been of dubious value. The difficulty has always been in finding an adequate definition for ideal multiplication. In every case so many important properties of ideals have failed to generalize that the theory is practically without substance. It would seem that the enlargement of the domain to permit unique factorization will have to be accomplished through other agencies than the Dedekind ideal.

2. *Domains of Integrity.* The elements of a rational algebra $\mathfrak{A}$ of order n are given by

* An address presented at the invitation of the program committee at the meeting of the Society in Minneapolis, September 10, 1931.

$$a_1\epsilon_1 + a_2\epsilon_2 + \cdots + a_n\epsilon_n,$$

where the basis numbers $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ are linearly independent with respect to the rational field $\mathfrak{R}$ and the a's range over $\mathfrak{R}$. The algebra is closed under multiplication, so that there exist numbers c_{ijk} of $\mathfrak{R}$, called the constants of multiplication, such that

$$\epsilon_i\epsilon_j = \sum_{k=1}^{n} c_{ijk}\epsilon_k, \qquad (i, j = 1, 2, \cdots, n).$$

We assume that multiplication is associative but not that it is commutative. We further assume that $\mathfrak{A}$ contains a principal unit or modulus.

It is evident that if $\epsilon_i = \theta^{i-1}$, where θ is a root of an irreducible equation of degree n, $\mathfrak{A}$ is an algebraic field. Thus the rational algebra is a natural extension of the algebraic field.

While the term *algebraic integer* is well defined, *integral number* of a rational algebra is not. The concept of *domain of integrity* is capable of generalization, however. An algebra $\mathfrak{A}$ may possess several such domains of integrity.

The most satisfactory definition for domain of integrity has been given by Dickson* by means of the five postulates:

R: For every element of the set, the coefficients of the rank equation are rational integers.

C: The set is closed under addition, subtraction and multiplication.

U: The set contains the modulus 1.

M: The set is maximal.

O: The set is of order n.

It is proved that if $\mathfrak{A}$ is not semi-simple,[2] it can be written as a sum of a semi-simple algebra and a nilpotent algebra, and that the arithmetic of $\mathfrak{A}$ is associated with that of the semi-simple component.[1] We shall therefore devote attention only to semi-simple algebras.

For a domain of integrity $\mathfrak{D}$ of a semi-simple rational algebra $\mathfrak{A}$, it is always possible to choose a basis $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ so that $\epsilon_1 = 1$, every number of $\mathfrak{D}$ is given by

* See the bibliography on p. 853; the reference here is to No. 1 of that list. Hereafter, references will be by a small numeral corresponding to the number in the list.

$$a_1\epsilon_1 + a_2\epsilon_2 + \cdots + a_n\epsilon_n,$$

where the a's are now rational integers, and such that the constants of multiplication c_{ijk} are likewise rational integers.

The general number satisfies an equation of minimum degree called the *rank equation*. The constant term of this equation is called the *norm* of the number. Numbers of norm 0 are called *divisors of zero*, those of norm ± 1 are called *units*. The norm of the product is equal to the product of the norms of the factors. Those numbers, neither divisors of zero nor units, which can be represented as a product of two numbers neither of which is a unit are called *composite*, and in the contrary case *prime*. Every composite number can be represented as a product of a finite number of primes, but not always uniquely.

Due to the failure of the commutative law, the concept of greatest common divisor must be replaced by the concepts of *greatest common right divisor* (g.c.r.d.) and *greatest common left divisor* (g.c.l.d.). We shall say that δ is a g.c.r.d. of α and β if there are numbers λ, μ of $\mathfrak{D}$ such that

$$\alpha = \lambda\delta, \quad \beta = \mu\delta, \tag{1}$$

and if also there are numbers ρ, σ of $\mathfrak{D}$ such that

$$\delta = \rho\alpha + \sigma\beta. \tag{2}$$

It is immediately evident that every c.r.d. of α and β divides δ. A similar definition holds for g.c.l.d. Of course two numbers α and β do not have to possess either a g.c.r.d. or a g.c.l.d.

3. *Ideals.* As a criterion for evaluating the various modifications of the Dedekind ideal theory which have been put forward in connection with the problem of unique factorization, we take the definition given by Dickson.[3] This demands of an ideal system merely the following properties:

(i) Each integer (not a divisor of zero) and its associates shall correspond to the same unique ideal.

(ii) There shall be a subset of principal ideals which shall be isomorphic with the numbers of $\mathfrak{D}$.

(iii) The ideal corresponding to the units shall be the identity element of multiplication and shall have no factor different from itself.

It will be interesting to see how far the ideal systems which have been advanced come from meeting these conditions.

Dedekind ideals, modified to fit non-commutative domains, may be defined as follows: A *left ideal** is an infinite set of numbers of $\mathfrak{D}$ which is closed under addition and subtraction, and closed under multiplication on the left by the numbers of $\mathfrak{D}$. If not every number of the ideal is a divisor of zero, the ideal is called *non-singular*.

The set of numbers

$$\xi_1\alpha_1 + \xi_2\alpha_2 + \cdots + \xi_k\alpha_k,$$

where the $\alpha_1, \cdots, \alpha_k$ are fixed and the $\xi_1, \cdots, \xi_k$ variable numbers of $\mathfrak{D}$, obviously constitutes a left ideal $(\alpha_1, \cdots, \alpha_k]$. In fact, there are no others. If $k=1$, the ideal is principal, written $\mathfrak{a}=(\alpha]$.

That Dedekind ideals are fundamentally connected with the problem of the g.c.d. is evident from the fact that *a necessary and sufficient condition that α and β have a g.c.r.d. is that the left ideal $(\alpha, \beta]$ be principal.* For if $(\alpha, \beta]$ is principal, there exists a number δ such that the sets

$$\xi\delta, \quad \eta\alpha + \zeta\beta, \qquad (\xi, \eta, \zeta \text{ variable}),$$

are equal. Thus in particular for $\eta=1$, $\zeta=0$ there is a number ξ_1 such that $\alpha=\xi_1\delta$, and for $\eta=0$, $\zeta=1$ there is a number ξ_2 such that $\beta=\xi_2\delta$. Furthermore, for $\xi=1$ there are numbers η_1, ζ_1 such that $\delta=\eta_1\alpha+\zeta_1\beta$. Thus δ is a g.c.r.d. of α and β. The converse is evident, for by (2) every number $\xi\delta$ is a linear combination of α and β, and by (1) every linear combination of α and β is a multiple of δ.

It is easily proved that every ideal has a basis $\omega_1, \omega_2, \cdots, \omega_n$ such that the numbers of the ideal are given by the linear combinations of these basis numbers with rational integral coefficients. An ideal is *non-singular* if any one of the following properties holds:

(a) The numbers $\omega_1, \omega_2, \cdots, \omega_n$ are linearly independent.
(b) The ideal contains rational integers.
(c) Not every element is a divisor of zero.
(d) The norm $|g_{rs}|$ is not zero, where $\omega_i=\sum g_{ij}\epsilon_j$.

* Probably it would have been more logical to have reversed the definitions of left and right ideals as was done by Hurwitz[4] for quaternions, for the essential property of $(\alpha]$ is not that it is closed under multiplication on the left, but that α is a right divisor of every number in it.

4. *The Early Literature.* In first introducing ideals Kummer was interested only in establishing the unique factorization law in certain algebraic fields. The existence of the g.c.d. *together with the commutative law* implies unique factorization. Without the commutative law it does not, as was shown by Hurwitz[4] in what seems to be the earliest use of ideals in a non-commutative algebra. Unique factorization does not hold for all integral quaternions, yet every quaternion ideal is principal, and every pair of integral quaternions has a g.c.r.d. and a g.c.l.d. unique up to unit factors.

From this example it is evident that too much is not to be expected from the ideal theory in the non-commutative case. In the generalization from algebraic fields to rational algebras, the problem of the greatest common divisor and the problem of unique factorization follow divergent paths, and the Dedekind ideal theory is fundamentally connected with the former problem.

Beginning about 1906, L. G. du Pasquier[5] wrote a series of papers investigating domains of integrity in rational algebras. He did not use Wedderburn's theory of linear algebras, however, and his work consisted principally of investigations of particular algebras. He recognized the advisability of working in maximal domains of integrity, and knew that maximal domains according to his definition of domain do not exist in all algebras. We now know that his difficulties were encountered with algebras which were not semi-simple. His conjecture that the ideal theory is helpless to introduce unique factorization into domains which are not maximal was definitely proved by Dickson.[3]

Probably the most important of the algebras considered in detail by du Pasquier[6] is the complete matric algebra over the rational field. He called a matrix *integral* if all its elements were rational integers. He developed a euclidean algorithm for such matrices and proved that a set of integral matrices $M_1, M_2, \cdots, M_k$, not all singular, possess a g.c.r.d. unique up to a left factor A which is integral and of determinant ± 1. Conversely, if D is a g.c.r.d., so is AD for every unimodular integral matrix A. Similar results hold for the g.c.l.d.

Du Pasquier defined right and left ideals for this algebra and showed that in virtue of the existence of the euclidean algo-

rithm every ideal not composed exclusively of divisors of zero is principal. He also discussed two-sided ideals and found that they are never singular.

5. *Speiser's Theory.* The first paper dealing exclusively with ideals in domains of linear algebras was written by Andreas Speiser[7] in 1926. The first part of the paper is devoted to ideals in commutative domains, that is, rings. In a maximal domain of integrity the ideal product is defined in the usual way as the totality of sums of products consisting of a number of the first ideal by a number of the second. Unique factorization into prime ideals holds. For the commutative case Speiser's results are interesting and very satisfactory.

For non-commutative domains of integrity, however, Speiser does not seem to have been so successful. A one-sided ideal, defined in the usual way, is determined by its residue classes, which in turn are represented by matrices with elements in a Galois field. The intersection of two ideals is called their least common multiple (l.c.m.). The ideal formed by adding every number of the first to every number of the second is called their g.c.d. If the g.c.d. of two ideals is the unit ideal, they are relatively prime, and their product is defined to be their l.c.m. Thus multiplication of prime ideals is commutative in a non-commutative domain of integrity. Dickson's second requirement (ii) for an ideal is violated.

If the norm of a right ideal $[\mathfrak{b}]$ is $p_1^{a_1}p_2^{a_2}\cdots p_l^{a_l}$, then $[\mathfrak{b}]$ can be written as $[q_1)[q_2)\cdots[q_l)$, where the factors are prime in pairs (so that their product is defined as above) and each is a divisor of a power of a two-sided prime ideal which divides the norm. The factors are commutative and define $[\mathfrak{b}]$ uniquely.

The factorization of each $[q)$ is treated somewhat after the manner of the series of composition of a finite group. Suppose that $[q)$ is an ideal divisor of p^a, where p is a natural prime. By adjoining p to $[q)$ there results an ideal divisor $[\mathfrak{P}_1)$ of p. The quotient of $[q)$ by $[\mathfrak{P}_1)$ is obtained by taking those residue classes in $[q)$ which are divisible by p and dividing them by p. This new ideal is called $[q_1)$. Now adjoin p to $[q_1)$ and obtain $[\mathfrak{P}_2)$ which is similarly a divisor of $[\mathfrak{P}_1)$. This process ends in a steps, and then by definition

$$[q) = [\mathfrak{P}_1)[\mathfrak{P}_2)\cdots[\mathfrak{P}_a).$$

Each factor is a divisor of the same two-sided prime ideal, and each divides the preceding. The factors are uniquely determined, but *the product is not in general uniquely determined by the factors.* Ideal multiplication is not uniquely defined.

Speiser regrets that a number of the most interesting concepts of algebraic number theory do not carry over. He mentions in particular the failure of the theorem that the ideal product can be defined through the products of the basis numbers. He also remarks upon the lack of a definition of ideal class, and the consequent vacuousness of the theorem that the class number is finite. He admits the belief that these concepts and theorems are essentially restricted to the algebraic number theory.

6. *Krull's Theory.* In a recent paper *Zur Theorie der zweiseitigen Ideale in nichtkommutativen Bereichen,*[8] the ideal theory is treated from a postulational point of view. The essential difficulty of a definition of ideal product is avoided by laying down a set of postulates which ideal multiplication is to obey.

This paper of Krull seems to touch only slightly upon ideals in rational algebras, so we shall not discuss it in detail. The concept of *halbkommutativ* is introduced for those domains in which every pair of prime ideals are commutative. The two-sided ideals are not isomorphic with the integral numbers, and the result which he emphasizes is that the ideals are not as a rule either commutative or *halbkommutativ.*

7. *Present Status of the Unique Factorization Problem.* This year Miss Grace Shover[9] in collaboration with the writer advanced a theory of ideal multiplication using integral matrices. Ideal multiplication is defined through the product of its basis numbers, but as Speiser noted, this definition is not as fruitful as one could wish. The norm of the product is not always equal to the product of the norms, and consequently Dickson's second requirement is not always complied with.

The conclusion seems to be that all attempts up to the present to apply Dedekind ideals to the problem of unique factorization in a non-commutative domain of integrity have been wrecked because of the lack of a satisfactory definition of ideal multiplication. Perhaps one is justified in hazarding the opinion that the theory will not extend.

8. *The Matric Theory.* In 1929 the writer[10] attempted to see

how far the ideal theory would extend without the use of ideal multiplication. It was surprising that so much of the ideal theory as regards the greatest common divisor carried over intact. The right and left class numbers were satisfactorily defined, and Miss Shover[11] has just proved them finite and equal. A necessary and sufficient condition that every pair of numbers of $\mathfrak{D}$ possess a g.c.r.d. unique up to a unit left factor is that the class number of $\mathfrak{D}$ be 1. In this case they also possess a g.c.l.d. unique up to a unit right factor. Moreover, a method for finding these greatest common divisors will be developed in §9.

The use of integral matrices in the theory of ideals in an algebraic field was not entirely new, but it had not been thoroughly exploited. Of course a correspondence between ideals and quadratic forms was known to Dedekind. Poincaré[12] used the matrix (g_{rs}) to represent the ideal with basis $\omega_i = \sum g_{ij}\epsilon_j$ but he did nothing with the matrix except to consider it as an aggregate of n^2 numbers.

Another result of Poincaré[13] was of more significance. He showed essentially that every associative algebra is isomorphic with a matric algebra, and it is but a step to see that when the basis numbers of the algebra form a basis for the domain of integrity $\mathfrak{D}$, the elements of every matrix corresponding to an integral number are rational integers, and conversely. Whether Poincaré knew that this correspondence could be extended to ideals is problematical.[14] His statements on this point have justly been termed "obscure."*

A correspondence between ideals and integral matrices different from that of Poincaré was set up by A. Chatelet.[15] Let an algebraic number θ satisfy an irreducible equation whose roots are $\lambda_1, \lambda_2, \cdots, \lambda_n$. Denote the diagonal matrix $(\delta_{rs}\lambda_s)$ by $[\lambda_i]$. There exists a matrix $T = A[\lambda_i]A^{-1}$ with rational integral elements if and only if the first column of A apart from a constant factor constitutes a basis for an ideal, the other columns of A being the conjugates of the first. By keeping A fixed and letting θ vary over the numbers of $\mathfrak{D}$, matrices corresponding to all numbers of the ideal are obtained. If A corresponds to the unit ideal, the corresponding T's represent the numbers of

* National Research Council, *Bulletin on Algebraic Numbers*, I, p. 26, footnote 4.

$\mathfrak{D}(\theta)$, and the representation is an isomorphism with respect to addition and multiplication.

Let D be a g.c.d. of the matrices representing the n numbers whose ideal g.c.d. we seek. Then DA has as its first column this base. Chatelet calls D an ideal factor. In a brief account in the Comptes Rendus[16] he states that "to the divisibility of ideals corresponds naturally the divisibility of ideal factors." Upon consulting a book[17] published by him in the following year it appears that his investigations had not been carried as far as one might be inclined to suppose from a reading of the former article. He does, however, give a method for calculating the product of two ideals.

9. *The Problem of the Greatest Common Divisor.* Since matrices are in general non-commutative, the matric theory seemed a natural tool to apply to ideals in non-commutative domains of integrity. Suppose we have a domain of the type described in §2 with basis $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$. To the number

$$\xi = x_1\epsilon_1 + x_2\epsilon_2 + \cdots + x_n\epsilon_n,$$

we make correspond its first matrix

$$R(\xi) = (\sum x_i c_{isr}).$$

The first matrices give a matric representation of the algebra. So do the second matrices

$$S(\xi) = (\sum x_i c_{ris}).$$

This is the correspondence of Poincaré.[13]

Let $\mathfrak{j}$ be a left ideal with basis $\omega_1, \omega_2, \cdots, \omega_n$, where

$$\omega_i = \sum g_{ij}\epsilon_j.$$

The matrix $G=(g_{rs})$ is said to *correspond* to the ideal. The matrices AG, where A is integral and unimodular, give all matrices corresponding to the same ideal.

The fundamental theorem[10] is that a necessary and sufficient condition that a matrix G correspond to an ideal is that there exist n integral matrices $D_1, D_2, \cdots, D_n$ such that

$$GR_i^{\mathrm{T}} = D_iG, \qquad (i = 1, 2, \cdots, n), \tag{3}$$

where R_i^{T} denotes the transpose of $R(\epsilon_i)$. Thus if G is non-

singular it transforms the transposed first matrix of every number of $\mathfrak{D}$ into an integral matrix.

It so happens that under this definition of correspondence the second matrix $S(\xi)$ corresponds to the principal left ideal $(\xi]$. Then (3) becomes

$$S(\xi)R^{T}(\eta) = R^{T}(\eta)S(\xi),$$

that is, the second matrices of every given number ξ are commutative with the transposed first matrices of every number η of $\mathfrak{D}$. The converse is also true, namely, that every integral matrix commutative with every $R^{T}(\eta)$ is the second matrix of some number of $\mathfrak{D}$.

The concept of equivalence of ideals is introduced in a manner analogous to that in algebraic number theory. If $\omega_1, \omega_2, \cdots, \omega_n$ constitute a basis for an ideal $\mathfrak{k}$, and if s is a number of $\mathfrak{D}$, then $\omega_1 s, \omega_2 s, \cdots, \omega_n s$ is a basis for an ideal which we denote by $\mathfrak{k}s$. This is essentially a definition of scalar multiplication for ideals. If $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are two non-singular ideals, they are called *equivalent* if there exist numbers s_1 and s_2 of $\mathfrak{D}$ such that $\mathfrak{k}_1 s_1 = \mathfrak{k}_2 s_2$. It is shown that if G_1 corresponds to $\mathfrak{k}_1$ and G_2 to $\mathfrak{k}_2$, a necessary and sufficient condition that $\mathfrak{k}_1$ and $\mathfrak{k}_2$ be equivalent is that the matrices D_{1i}, D_{2i} defined by

$$G_1 R_i^{T} = D_{1i} G_1, \quad G_2 R_i^{T} = D_{2i} G_2,$$

be connected by a relation

$$D_{1i} = A D_{2i} A^{-1}, \qquad (i = 1, 2, \cdots, n),$$

where A is integral and unimodular.

The (left) class number h is defined as the number of non-equivalent non-singular ideals in $\mathfrak{D}$. A necessary and sufficient condition that every pair of numbers in $\mathfrak{D}$ possess a g.c.r.d. unique up to a unit left factor is that this class number be 1. The left and right class numbers are finite and equal.[11]

One of the interesting by-products of the theory is the proof that every ideal class possesses a basis composed of n integral matrices $B_1, B_2, \cdots, B_n$ such that every matrix corresponding to an ideal of the class is of the form AG, where A is integral and unimodular and G is of the form

$$h_1 B_1 + h_2 B_2 + \cdots + h_n B_n,$$

where the h's are rational integers. If the class is non-singular, that is, contains at least one non-singular ideal, the B's are linearly independent. Every ideal of the class is given by this form not once but exactly p times, where p is the (cardinal) number of units in $\mathfrak{D}$.

It is also of interest that the matric theory is able to provide for all domains of integrity $\mathfrak{D}$ of semi-simple rational algebras an effective euclidean algorithm, something which has been developed previously for only a few algebraic fields. Let α and β be two numbers of $\mathfrak{D}$ not divisors of zero. Let P be a matric g.c.r.d. of $S(\alpha)$ and $S(\beta)$. Then P corresponds* to a basis of the ideal $(\alpha, \beta]$ composed of the numbers $\xi\alpha+\eta\beta$, and consequently the matrices

$$PR_i^{\mathsf{T}}P^{-1} = T_i, \qquad (i = 1, 2, \cdots, n),$$

are all integral.

Now $(\alpha, \beta]$ is principal if and only if there exists a unimodular integral matrix A such that $AP=S(\delta)$, where δ is a g.c.r.d. of α and β. Since every second matrix is commutative with R_i^{T}, we can find whether A exists, and determine it if it does exist, from the relations $APR_i^{\mathsf{T}} = R_i^{\mathsf{T}}AP$, which are equivalent to

$$R_i^{\mathsf{T}}A = AT_i, \qquad (i = 1, 2, \cdots, n).$$

By determining A^{A} (adjoint of A) we determine A. But

$$A^{\mathsf{A}}R_i^{\mathsf{T}} = T_iA^{\mathsf{A}}, \qquad (i = 1, 2, \cdots, n), \tag{4}$$

so the matrices A^{A} constitute a minor class of ideal matrices.[10] This class has a basis $B_1, B_2, \cdots, B_n$ such that every matrix satisfying (4) is of the form

$$x_1B_1 + x_2B_2 + \cdots + x_nB_n.$$

Hence the n^3 equations resulting from (4) are in fact equivalent to but $n(n-1)$ equations in n parameters provided the matrix A^{A} exists. The condition $A^{\mathsf{A}} = \pm 1$ leads to the generalized Pell equation

$$|x_1B_1 + x_2B_2 + \cdots + x_nB_n| = \pm 1,$$

* Proved by the writer for algebraic fields but readily extensible to domains of integrity $\mathfrak{D}$.[18]

the solution of which is the same type of problem as the determination of the units of $\mathfrak{D}$. This was to have been expected, since δ is unique only up to a unit factor. Every determination of A leads to a determination of δ.

Since

$$AP = x_1S_1 + x_2S_2 + \cdots + x_nS_n,$$

the x's are readily found by the method of undetermined coefficients. By the same method we can find the numbers λ, μ of $\mathfrak{D}$ such that $\delta = \lambda\alpha + \mu\beta$.

I hope to convince you of the practicability of this generalized euclidean algorithm in a future paper.

10. *Conclusion.* In some respects the attempts to extend the ideal theory to non-commutative domains of integrity have failed, for the unique factorization problem has not been solved. On the other hand, much light has been thrown on the structure of systems of ideals, and on the relationship between the unique factorization problem and the problem of the greatest common divisor.

Then, too, the attempts to solve this problem have led to the use of new instruments in the ideal theory, and these in turn have led to the discovery of new results in algebraic number theory—the basis of an ideal class, the generalized euclidean algorithm, and direct methods for multiplying ideals[9] and for finding the canonical basis of an ideal.[18] A new connection between integral matrices and ideals, in no way dependent upon the preceding results, is the following theorem of C. G. Latimer.[19] If $p(x) = 0$ is the irreducible equation defining the integral number θ, the class number of $\mathfrak{F}(\theta)$ is the number of dissimilar integral matrices satisfying $p(x) = 0$.

In closing, I rather timidly put forward the suggestion that perhaps the unique factorization problem has been solved for some time and that this fact has not been properly appreciated. We have been so intent on generalizing Dedekind ideals that we have refused to consider other possible solutions. Let us consider the domain of integrity $\mathfrak{D}'$ composed of *all* integral matrices of order n. This domain of integrity contains a subdomain, consisting of the second matrices of the numbers of $\mathfrak{D}$, which is isomorphic with $\mathfrak{D}$ not only under multiplication but under addition as well. All of the units of $\mathfrak{D}$ are contained among

the units of $\mathfrak{D}'$, all the divisors of zero of $\mathfrak{D}$ are contained among the divisors of zero of $\mathfrak{D}'$. The class number of $\mathfrak{D}'$ is 1, and every number of $\mathfrak{D}'$ can be expressed as a product of prime matrices[6] in one and only one way apart from unit factors. Is there any reason for pursuing this problem further?

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A NOTE ON PRIMITIVE IDEMPOTENT ELEMENTS OF A TOTAL MATRIC ALGEBRA*

BY F. S. NOWLAN

We consider a total matric algebra M over a field F, whose general element is

$$u = \sum \alpha_{ij} e_{ij}, \qquad (i, j = 1, \cdots, n),$$

where $e_{ij}e_{lk} = e_{ik}$ if $j = l$, and $e_{ij}e_{lk} = 0$ for $j \neq l$.

THEOREM 1. *A necessary and sufficient condition that $u = \Sigma \alpha_{ij} e_{ij}$ be idempotent in M is*

$$\sum_s \alpha_{ps}\alpha_{sq} = \alpha_{pq}, \qquad (p, q = 1, \cdots, n). \tag{1}$$

This is seen immediately on writing

$$u^2 = \sum_{p,q,s} \alpha_{ps}\alpha_{sq} e_{pq}$$

and comparing with

$$u = \sum_{p,q} \alpha_{pq} e_{pq}.$$

THEOREM 2. *A necessary and sufficient condition for an idempotent element u to be primitive in M is*

$$\alpha_{pi}\alpha_{jq} = \alpha_{pq}\alpha_{ji}, \qquad (p, q, i, j = 1, \cdots, n). \tag{2}$$

For, let

$$u = \sum_{r,t} \alpha_{rt} e_{rt}$$

be a primitive indempotent element of M. Let $\alpha_{ji} \neq 0$. Then the element $ue_{ij}u/\alpha_{ji}$ is idempotent in uMu, since

$$\left(\frac{ue_{ij}u}{\alpha_{ji}}\right)^2 = \frac{u \cdot e_{ij}ue_{ij} \cdot u}{\alpha_{ji}^2} = \frac{u\alpha_{ji}e_{ij}u}{\alpha_{ji}^2} = \frac{ue_{ij}u}{\alpha_{ji}}.$$

Hence† we have $ue_{ij}u/\alpha_{ji} = u$, and $ue_{ij}u = \alpha_{ji}u$. Equating coef-

* Presented to the Society, June 15, 1927.

† Dickson, *Algebras and their Arithmetics*, p. 55.

ficients of e_{pq}, we have $\alpha_{pi}\alpha_{jq}=\alpha_{pq}\alpha_{ji}$. Conversely, suppose that

$$u=\sum\alpha_{rt}e_{rt}$$

is idempotent in M and let

$$\alpha_{pi}\alpha_{jq}=\alpha_{pq}\alpha_{ji}, \qquad (p, q, i, j=1, \cdots, n).$$

It follows that u is a primitive idempotent. In proof, let

$$m=\sum\beta_{sp}e_{sp}$$

be any element of M. Then

$$umu = \sum_{r,s,p,t}\alpha_{rs}\beta_{sp}\alpha_{pt}e_{rt} = \sum_{r,s,p,t}\alpha_{ps}\beta_{sp}\cdot\alpha_{rt}e_{rt}.$$

The coefficient of $\alpha_{rt}e_{rt}$ is

$$\sum_{s,p}\alpha_{ps}\beta_{sp},$$

which is independent of r and t. It follows that

$$umu = \left(\sum_{s,p}\alpha_{ps}\beta_{sp}\right)\left(\sum_{r,t}\alpha_{rt}e_{rt}\right) = \sum_{s,p}\alpha_{ps}\beta_{sp}\cdot u.$$

This shows that the algebra uMu is of order 1, based on its modulus.* Hence u is a primitive idempotent element of M.

Applying (2) to (1), we obtain the following corollary.

COROLLARY. *Relations* (2) *and* (3) *below constitute necessary and sufficient conditions for u to be a primitive idempotent element of M:*

$$\left.\begin{aligned} &(2) && \alpha_{pi}\alpha_{jq} = \alpha_{pq}\alpha_{ji} \\ &(3) && \textstyle\sum\alpha_{pp} = 1 \end{aligned}\right\}, \qquad (p, q, i, j = 1, \cdots, n).$$

The above Corollary leads to a simple device for obtaining the coordinates of a primitive idempotent element of a total matric algebra of order n. Construct a square array of n rows and n columns, as follows.

First write numbers, $\alpha_{ii}(i=1, \cdots, n)$ along the principal diagonal so that $\Sigma\alpha_{ii}=1$. Then write in numbers α_{i1} arbitrarily in the first column. Next write in numbers α_{ij} chosen to satisfy the relation $\alpha_{i1}\alpha_{jj}=\alpha_{ij}\alpha_{j1}$, $(i=1, \cdots, n;\ j=2, \cdots, n)$. It follows readily that $\alpha_{ij}\alpha_{km}=\alpha_{im}\alpha_{kj}$, $(i, j, k, m=1, \cdots, n)$.

We illustrate with the array

* Nowlan, *On the direct product of a division and a total matric algebra*, this Bulletin, vol. 36 (1930), p. 267, Theorem 5.

$$\begin{pmatrix} \frac{1}{3} & 1 & 0 \\ \frac{2}{9} & \frac{2}{3} & 0 \\ -2 & -6 & 0 \end{pmatrix}$$

for the case $n=3$. This gives the primitive idempotent element

$$u = \tfrac{1}{3}e_{11} + e_{12} + \tfrac{2}{9}e_{21} + \tfrac{2}{3}e_{22} - 2e_{31} - 6e_{32}.$$

We define supplementary primitive idempotent elements as follows.

DEFINITION. A set of primitive idempotent elements is said to be *supplementary* in case their sum equals the modulus and if, further, the product of each pair in either order is zero.*

We now determine necessary and sufficient conditions that a set of primitive idempotent elements shall be supplementary.

Let u_i and $u_j (i \neq j)$ be two of a set of supplementary primitive idempotent elements. A necessary and sufficient condition for the relation $u_i u_j = u_j u_i = 0$, is obviously

$$\sum_s \alpha_{rs}^{(i)} \alpha_{st}^{(j)} = 0,$$

where $\alpha_{rs}^{(i)}$ and $\alpha_{st}^{(j)}$ are the general coordinates of u_i and u_j respectively. Combining this result with the condition that the sum of the components of a set of supplementary primitive elements shall equal the modulus, we have the following result.

THEOREM 3. *Equations* (4), (5) *and* (6), *which follow, constitute necessary and sufficient conditions that a set of primitive idempotent elements shall be supplementary*:

$$\sum_s \alpha_{rs}^{(i)} \alpha_{st}^{(j)} = 0, \qquad (i \neq j \text{ and } r, s, t, i, j = 1, \cdots, n); \tag{4}$$

$$\sum_{k=1}^{n} \alpha_{rl}^{(k)} = 0, \qquad (r \neq l \text{ and } r, l = 1, \cdots, n); \tag{5}$$

$$\sum_{k=1}^{n} \alpha_{rr}^{(k)} = 1, \qquad (r = 1, \cdots, n). \tag{6}$$

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* A supplementary set of primitive idempotent elements of a total matric algebra of order n^2 contains exactly n elements. See F. S. Nowlan, this Bulletin, vol. 36 (1930), p. 268.

REPRESENTATION OF A GROUP AS A TRANSITIVE PERMUTATION GROUP

BY G. A. MILLER

Let G be any group of finite order g and let H be any subgroup of order h contained in G. If the operators of G are separated into right or into left augmented co-sets with respect to H and if these $g/h=n$ co-sets are then multiplied successively on the right or on the left respectively by the various operators of G, they will be permuted as units according to a transitive permutation group T which is simply isomorphic with the quotient group of G with respect to the largest invariant subgroup of G which appears in H, if H is not itself invariant under G. If H is invariant under G, then T will be a regular group which is simply isomorphic with G/H. The case when H is non-invariant under G and does not involve any invariant subgroup of G besides the identity is especially important since T is then simply isomorphic with G, as was pointed out for right co-sets by W. Dyck in 1883.

If K is any subgroup of G which has operators in each of the co-sets of G with respect to H and if K_0 is the cross-cut of H and K, then K_0 may be invariant under K or it may involve an invariant subgroup under K. If one of these conditions is satisfied, H must involve a subgroup which is invariant under G and includes this invariant subgroup under K. This follows directly from the facts that this invariant subgroup is transformed into all of its conjugates under G by operators of H and that a complete set of conjugate subgroups always generates an invariant subgroup if it does not generate the entire group. We have then the following result.

THEOREM 1. *If a group G is separated into co-sets with respect to a subgroup H and if another subgroup K has operators in each of these co-sets, then the largest invariant subgroup under K which appears in the cross-cut of H and K is contained in an invariant subgroup of G which is found in H.*

In particular, when T is simply isomorphic with G, then the largest invariant subgroup under K which appears in the cross-cut of H and K is the identity.

A necessary and sufficient condition that a subgroup of G corresponds to a transitive subgroup of degree n in T is that it has operators in each of the co-sets of G with respect to H. In the case when T is simply isomorphic with G this transitive subgroup of degree n must also be simply isomorphic with the corresponding subgroup of G. We have then the following result.

THEOREM 2. *When G is represented as a simply isomorphic transitive permutation group of degree n with respect to a subgroup H, then a necessary and sufficient condition that a given subgroup of G corresponds to a simply isomorphic transitive subgroup of degree n is that operators of this subgroup of G appear in each of the co-sets of G with respect to H.*

When T is not simply isomorphic with G, a subgroup of G is not necessarily simply isomorphic with the corresponding subgroup of T. A necessary and sufficient condition that such a simple isomorphism exists is that this subgroup of G has only the identity in common with the largest invariant subgroup of G which is found in H. When H is invariant under G, then the regular group T will correspond to every subgroup of G whose operators are distributed among all of the co-sets of G with respect to H. Subgroups of G whose operators are not thus distributed will correspond to regular constituents of the subgroups of T. A necessary and sufficient condition that a subgroup of G corresponds either to a regular subgroup of T or to a regular constituent of a subgroup of T is that its cross-cut with H is invariant under this subgroup.

Suppose that G involves a subgroup H whose operators are distributed among all except one of the co-sets of G with respect to H and that T is simply isomorphic with G. The subgroup H must correspond to a simply isomorphic subgroup of T which is of degree $n-1$ and transitive on these $n-1$ letters. This subgroup must therefore appear in a conjugate of the subgroup of T which corresponds to H. It therefore results that T is multiply transitive. Moreover, when T is multiply transitive, G must contain such a subgroup.

THEOREM 3. *A necessary and sufficient condition that a group G appears as a multiply transitive group when it is represented as a transitive permutation group with respect to a subgroup H is that at least one subgroup of G has its operators distributed among all except one of the co-sets of G with respect to H.*

From this theorem it results directly that a necessary and sufficient condition that a group is r-fold transitive when it is represented as a simply isomorphic transitive group with respect to a subgroup H is that one can find $r-1$ successive subgroups $H_1, H_2, \cdots, H_{r-1}$ each of which after the first is contained in the preceding and has its operators distributed among all except one of the co-sets of G with respect to H in which the operators of the preceding subgroup are found, the operators of H_1 appearing in all these co-sets except one. It may be noted that the operators of all of these H's have the property that their products into any co-set which contains no operator of the corresponding H appear in this co-set.

Suppose that all the operators of G are distributed successively with respect to a subgroup H of G both into right and also into left co-sets. It is easy to verify that every such right co-set is identical with some left co-set of G with respect to a conjugate of H and that the totality of the right co-sets of H includes left co-sets of G with respect to all the conjugates of H under G. Similarly, the totality of the left co-sets with respect to H includes right co-sets with respect to all the conjugates of H under G. This results directly from the theorem that the multiplying operators of co-sets can always be so chosen that the totality of the right multipliers is identical with the totality of left multipliers. This proves the following theorem.

THEOREM 4. *The right co-sets of any group G with respect to a given subgroup H are composed of left co-sets of G with respect to all the conjugates of H under G, and vice versa.*

If more than one right co-set of G with respect to H is equal to a left co-set with respect to the same conjugate of H, the number of such right co-sets is equal to the index of H under the largest subgroup of G in which H is invariant. In particular, this number is an invariant of G. A necessary and sufficient condition that only one right co-set of G with respect to H is equal to a left co-set with respect to a given conjugate of H is that H is transformed into itself only by its own operators under G. This is also a necessary and sufficient condition that only one left co-set of G with respect to H is equal to a right co-set with respect to a given conjugate of H. The operators of G which when multiplied on the right into a given right co-set of G with respect to H have all their products in this co-set constitute the

conjugate of H in the equivalent left co-set, and vice versa.

A necessary and sufficient condition that T is simply transitive is that at least one conjugate of H under G has its operators distributed among less than $n-1$ co-sets of G with respect to H. In particular, when these operators are distributed among $n-2$ such co-sets n must be even and T must involve a system of imprimitivity composed of $n/2$ sets of letters. This is also a necessary and sufficient condition that H is invariant under a subgroup of G whose order is exactly $2h$. If G is simply isomorphic with T and a subgroup of G has its operators distributed among $n-2$ of the co-sets of G with respect to H, its order cannot exceed $2h$, and when it has this order it must involve a subgroup of index 2 which is conjugate with H under G. Moreover, H corresponds to a transitive subgroup of degree $n-2$ in T.

THE UNIVERSITY OF ILLINOIS

A NOTE ON TRANSFINITE ORDINALS

BY BEN DUSHNIK

In a supplementary note to an article of theirs,* Alexandroff and Urysohn demonstrated the following theorem.

If to every ordinal α of the second class there corresponds an ordinal $\mu(\alpha)$ such that $\mu(\alpha)<\alpha$, then there exists a non-denumerable set of ordinals of the second class

$$\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots, \alpha_\omega, \cdots, \alpha_\lambda, \cdots$$

such that

$$\mu(\alpha_1) = \mu(\alpha_2) = \cdots = \mu(\alpha_\lambda) \cdots .$$

The present note applies a different method to prove the following more general result.

THEOREM. *Let Ω_δ be the smallest ordinal whose power is $\aleph_\delta$, where $\delta>0$ is a non-limiting ordinal. If to every transfinite ordinal $\alpha<\Omega_\delta$ there corresponds an ordinal $\mu(\alpha)$ such that $\mu(\alpha)<\alpha$,*

* *Mémoire sur les espaces topologiques compacts*, Verhandelingen of the Amsterdam Academy, (1), vol. 45, No. 1.

then there will be a set of power $\aleph_\delta$ of transfinite ordinals $<\Omega_\delta$ such that the corresponding $\mu(\alpha)$ are all equal.

PROOF. Consider, for any transfinite $\alpha<\Omega_\delta$, the series

$$\alpha, \mu(\alpha), \cdots, \mu^i(\alpha), \cdots,$$

where $\mu^i(\alpha)$ is the ordinal corresponding to $\mu^{i-1}(\alpha)\,[i\geqq 2]$. By hypothesis

$$\alpha > \mu(\alpha) > \mu^2(\alpha) > \cdots;$$

since every descending series of ordinals is finite, and since $\mu^{i+1}(\alpha)$ is well-defined if $\mu^i(\alpha)$ is a transfinite ordinal, there must exist a finite number $m(\alpha)$ such that $\mu^{m(\alpha)}(\alpha)=n(\alpha)$, a finite number.

There is thus associated a definite finite number $n(\alpha)$ with every transfinite ordinal $\alpha<\Omega_\delta$. If for every finite n the corresponding set of ordinals α for which $n(\alpha)=n$ has the power $\aleph^{(n)}\leqq\aleph_{\delta-1}$, then the set of all ordinals $<\Omega_\delta$ would have the power

$$\sum_{i=0}^{\infty}\aleph^{(i)} \leqq \aleph_0\cdot\aleph_{\delta-1} < \aleph_\delta,$$

which is absurd. Hence there must exist a set K'' of transfinite ordinals $\alpha<\Omega_\delta$, such that the power of K'' is $\aleph_\delta$ and such that if α belongs to K, then $\mu^{m(\alpha)}(\alpha)=N$, a fixed finite number. Similarly, there will exist a set K' contained in K'', of the same power as K'', such that if α belongs to K', then $m(\alpha)=M$, a fixed finite number.

The ordinals of the set K' having been arranged in a well-ordered series (say, of type Ω_δ),

$$\alpha_1, \alpha_2, \cdots, \alpha_\omega, \cdots, \alpha_\beta, \cdots,$$

consider the matrix of M columns in which the row of index β contains the elements

$$\alpha_\beta, \mu(\alpha_\beta), \cdots, \mu^M(\alpha_\beta),$$

in the order as given. Since in the first column of the matrix there are $\aleph_\delta$ different ordinals, while in the last column the same number N is repeated, there must be a column of index $q<M$ in which appear $\aleph_\delta$ distinct ordinals, such that in the $(q+1)$th column there will appear at most $\aleph_{\delta-1}$ distinct ordi-

nals. As before, one can obtain from the qth column of the matrix at least one set K of $\aleph_\delta$ distinct ordinals $\mu^q(\alpha)$ for which all the $\mu^{q+1}(\alpha)$ are identical.

Obviously, this set K is the one whose existence was to be proved.

It may be noted that the Zermelo Axiom has been used freely in the above discussion. Further, it should be mentioned that the method of Alexandroff and Urysohn will also suffice to prove the theorem stated here; however, it is hoped that the method of proof presented here is not without interest.

Finally, the specification in the theorem that δ be a non-limiting number is essential, as is indicated by the following example.

Let $\delta=\omega$, the smallest transfinite ordinal, and consider the set of all the transfinite ordinals $\alpha<\Omega_\omega$, the power of the set being thus $\aleph_\omega$. If $\omega\leqq\alpha<\Omega_1$, we put $\mu(\alpha)=1$; in general, for $n\geqq 1$, if $\Omega_n\leqq\alpha<\Omega_{n+1}$, we set $\mu(\alpha)=\Omega_{n-1}$, where $\Omega_0=\omega$. Except that δ is now a limiting number, all the other conditions of the theorem are satisfied; yet the conclusion no longer holds. Indeed, $\mu(\alpha)$ is equal to one of the numbers

$$1, \omega, \Omega_1, \Omega_2, \cdots, \Omega_n, \cdots,$$

while the set of α's for which, say, $\mu(\alpha)=\Omega_n$ has the power $\aleph_{n+2}<\aleph_\omega$.

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ON THE APPLICATION OF A THETA FORMULA TO REPRESENTATION IN BINARY QUADRATIC FORMS*

BY GORDON PALL†

In his remarkable 1885 dissertation,‡ *Applications of the Theory of Elliptic Functions to the Theory of Numbers*, Nazimoff uses the formula (1) to derive the number of representations in x^2+3y^2 and x^2+5y^2. Some of his theorems are not quite correct. He does not derive the series representing $\sum\sum q^{x^2+7y^2}$ since "the derivation is long and not difficult." He states that the method has only limited application, since generally in (4) the A's are functions of k.

It is the purpose of this paper to give the essential details for x^2+7y^2, and to prove that the only cases ax^2+by^2, ab odd, which are actually solvable by Nazimoff's method are $a=1$, $b=1$, 3, 5, 7. The final result for x^2+7y^2 is of course far from new.

At the time this paper was written the writer intended to examine the products of series involved in other cases than $a=1$, $b=1$, 3, 5, 7, in the hope of obtaining information on $N(n=x^2+11y^2)$ or other cases of several classes in a genus. But since then an arithmetic method has been discovered of finding a simple formula for the number of representations in any positive, binary quadratic form. It is possible, after the details of this theory have been worked out, that there may be applications to elliptic and modular functions.

The formula which Nazimoff uses is

$$\sum_{x,y} q^{ax^2+by^2} = (ab)^{-1/2}\phi(2K/(ab)), \tag{1}$$

where x, y run through all integers, a and b are given positive, odd, relative-prime integers, and

$$\phi(u) = \frac{2K}{\pi} \prod_{l=1}^{(a-1)/2} t(lbu) \prod_{h=1}^{(b-1)/2} t(hau). \tag{2}$$

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* Presented to the Society at Pasadena, November 28, 1931.

† National Research Fellow, California Institute of Technology.

‡ Translation by Arnold E. Ross, pp. 5–12.

Here $t(u)$ is an abbreviation for the elliptic function

$$\frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} = \frac{\vartheta_1(x)\vartheta_3(x)}{\vartheta_2(x)\vartheta_4(x)}, \qquad (u = 2Kx). \tag{3}$$

Thus $N(n=ax^2+by^2)$ is the coefficient of q^n in the expansion in powers of q of the value for $x=1/(ab)$ of the elliptic function $(ab)^{-1/2}\phi(u)$. We shall see in a moment that the latter, with $t(u)$, has the periods $u=2K$ and $2iK'$, and, in many cases, only simple poles. When this is so we can express $\phi(u)$ as a sum of simple theta-quotients with coefficients independent of u. For example,

$$\phi(u) = \sum_{\alpha} A_\alpha \left\{ \frac{\vartheta_1'(x-\alpha)}{\vartheta_1(x-\alpha)} + \frac{\vartheta_1'(\alpha)}{\vartheta_1(\alpha)} \right\}, \tag{4}$$

the summation being extended to a representative set of poles $x=\alpha$, and the coefficients A being the residues at these poles. When all the A's are absolute constants, independent of k, our problem can be solved, since we have the formula

$$\frac{\vartheta_1'(x)}{\vartheta_1(x)} = \operatorname{ctn} \pi x + 4 \sum_{r=1}^{\infty} \frac{q^{2r}}{1-q^{2r}} \sin 2\pi r x, \tag{5}$$

with which to form a comparison power series in q.

We use the notations of Tannery and Molk, whence

$$u = 2Kx, \quad \tau = iK'/K, \quad q = \exp \pi i \tau. \tag{6}$$

Hence $t(hau)$, $t(lbu)$ have the periods $x=1, \tau$, and $\phi(u)$ also has these periods. The poles of $t(hau)$ are the zeros of the denominator $\vartheta_2(hax)\vartheta_4(hax)$, namely

$$x = \frac{1}{2ah}(m_1 + m_2\tau), \tag{7}$$

where m_1, m_2 are any integers of opposite parities; the zeros of $t(hau)$ are given by (7) with $m_1 \equiv m_2 \pmod 2$. These poles and zeros are simple. Hence, if p is a positive integer, the number α such that

$$\alpha = (m_1 + m_2\tau)/(2p), \quad (m_1, m_2, p) = 1, \quad m_1 \not\equiv m_2 \pmod 2, \tag{8}$$

is a zero or pole of $t(hau)$ if and only if $p \mid ha$, and is the former

or the latter according as ha/p is even or odd. The number of multiples of p in the sequence

$$a, 2a, \cdots, \tfrac{1}{2}(b-1)a,$$

with an odd quotient for p cannot exceed the number with an even quotient by more than unity. Hence each product Π in (2) has only simple poles. If they have poles in common, $\phi(u)$ has a double pole. This occurs, for example, with $p=1$ if $a \equiv b \equiv 3 \pmod 4$. For no pole α is $\vartheta_1(\alpha)=0$.

The poles of $\phi(2Kx)$ in the parallelogram $P \equiv (0, 1, \tau, 1+\tau)$ are easily found by the preceding method. If $a=1$, $b=7$ they are seen at once to be $\rho+\sigma\tau$ and $\sigma+\rho\tau$, where

$$(9)\quad (\rho, \sigma) = (\tfrac{1}{2}, 0), (\tfrac{1}{4}, 0), (\tfrac{1}{4}, \tfrac{1}{2}), (\tfrac{3}{4}, 0), (\tfrac{3}{4}, \tfrac{1}{2}), (\tfrac{1}{6}, 0), (\tfrac{1}{6}, \tfrac{1}{3}), (\tfrac{1}{6}, \tfrac{2}{3}), (\tfrac{1}{2}, \tfrac{1}{3}), (\tfrac{1}{2}, \tfrac{2}{3}), (\tfrac{5}{6}, 0), (\tfrac{5}{6}, \tfrac{1}{3}), (\tfrac{5}{6}, \tfrac{2}{3}).$$

Let us calculate the residue A of $\phi(2Kx)$ at a simple pole $x=\alpha$ of the form (8). If p divides ha with an odd quotient, the limit as $x \to \alpha$ of

$$(x-\alpha) \cdot 2K \cdot t(ha \cdot 2Kx)$$

is readily seen to be $(-1)^{m_2+1}/(ha)$. If the quotient is even, the limit as $x \to \alpha$ of $t(ha \cdot 2Kx)/((x-\alpha)2K)$ is simply ha. Write $\beta = 2K\alpha$. By (4) and (5) and the preceding remarks, we find

$$(10)\quad \begin{aligned} A &= \lim \pi(x-\alpha)\phi(u) = \lim 2K(x-\alpha)\prod_l t(lbu)\prod_h t(hau) \\ &= Q\prod_l{}' t(lb\beta)\prod_h{}' t(ha\beta), \end{aligned}$$

where the primes indicate that the factors for which $p \mid lb$ or $p \mid ha$ are omitted, and Q is the product of the proper numerical factors $(-1)^{m_2+1}/(ha)$, ha, $(-1)^{m_2+1}/(la)$, la for these omitted values h and l.

Now we readily verify that

$$(11)\quad t(u+iK')t(u) = 1 = -t(u)t(u+K),$$

$$(12)\quad t(u+2iK') = t(u) = t(u+2K),$$

$$(13)\quad t(\tfrac{1}{2}K) = 1, \quad t(\tfrac{1}{2}iK') = i.$$

Hence, if $p\beta = (2m+1)K + 2niK'$, m, n being integers,

$$(14)\quad t(r\beta)t(\overline{p-r}\beta) = 1 = -t(r\beta)t(\overline{p+r}\beta),$$

$$t(\tfrac{1}{2}p\beta) = (-1)^m. \tag{15}$$

If $p\beta = 2mK + (2n+1)iK'$,

$$t(r\beta)t(\overline{p-r}\beta) = -1 = -t(r\beta)t(\overline{p+r}\beta), \tag{16}$$

$$t(\tfrac{1}{2}p\beta) = (-1)^n i. \tag{17}$$

Hence in the products $\prod'$ in (10), two factors the sum or difference of whose arguments is $p\beta$ multiplied by an odd integer combine into a constant independent of k, and similarly for single factors $t(\frac{1}{2}ps\beta)$, s odd.

Let $a=1$, $b\geqq 9$. Consider the pole $\alpha = 1/(b-3)$, whence $p=\frac{1}{2}(b-3)$, $m_1=1$, $m_2=0$. Except for a constant factor independent of k, the residue A is, by (14) and (15),

$$t(\tfrac{1}{2}(b-1)2K\alpha).$$

This actually depends on k, since it reduces to 1 if $k=1$, and to $\tan(\pi/(b-3))$ if $k=0$.

Let $b>a>1$, and consider $\alpha = 1/(2p)$, $p=\frac{1}{2}(b-1)a$. The second factor Π' in (10) is independent of k as above, but $\Pi_l t(lb\beta)$ is seen to depend on k, since it has the value 1 if $k=1$, and the value $\Pi_l \tan(lb\pi/((b-1)a)) \neq 1$ if $k=0$.

Let $a=1$, $b=3$. Then $\phi = (2K/\pi)t(u)$. The poles in P are given by $p=1$, $m_1=1$ or 0, $m_2=0$ or 1, and the residues A are respectively -1 and $+1$ by the rule for forming Q.

Let $a=1$, $b=5$. Then $\phi = (2K/\pi)t(u)t(2u)$. Only $p=2$ defines poles; and with it m_1 is 0 or 2, m_2 is 1 and 3, and vice-versa. By (15) and (10) the residues A for m_1 odd are $\frac{1}{2}(-1)^{(m_1-1)/2}$. By (10) and (17), $A = +\frac{1}{2}i(-1)^{(m_2-1)/2}$ if m_2 is odd.

We can easily derive Nazimoff's series for x^2+3y^2 and x^2+5y^2.

Let $a=1$, $b=7$, $\phi(u) = (2K/\pi)t(u)t(2u)t(3u)$. If $p=1$, each Π' in (1) is trivially unity, and

$$A = Q = (-1)^{m_2+1}\cdot 2\cdot(-1)^{m_2+1}/3 = \tfrac{2}{3}$$

in all cases. If $p=2$, $Q=(-1)^{m_2+1}/2$, and $t(\beta)t(3\beta) = (-1)^{m_2+1}$ by (14) and (16). Hence $A=\frac{1}{2}$ in all cases. If $p=3$, $Q=(-1)^{m_2+1}/3$, and $t(\beta)t(2\beta) = (-1)^{m_2}$ by (14) and (16). Hence $A=-\frac{1}{3}$.

It follows that this method gives an immediate solution to the problem of finding the formula for $N(n=ax^2+by^2)$ only if $a=1$, $b=1, 3, 5, 7$.

Write $x=1/(ab)$, $\alpha=\rho+\sigma\tau$. On combining (4) and (5), we have to calculate

$$(18)\quad \sum_{\alpha} A_\alpha\{\sin 2\pi n\alpha+\sin 2\pi n(x-\alpha)\}=(1-\cos 2\pi nx)s_n+(\sin 2\pi nx)c_n,$$

$$s_n = \sum_{\alpha} A_\alpha \sin 2\pi n\alpha, \quad c_n = \sum_{\alpha} A_\alpha \cos 2\pi n\alpha,$$

where the summation is extended to the values α in P. Write

$$(19)\qquad t_n = \sum{}' A_\alpha e^{2n\rho i\pi} q^{2n\sigma},$$

where the prime indicates the omission of the values $\alpha=\rho+\sigma\tau$ for which $\sigma=0$. Then

$$(20)\quad \sum_{\alpha} A_\alpha\{\operatorname{ctn}\pi\alpha+\operatorname{ctn}\pi(x-\alpha)\} = 2i\sum_{n=1}^{\infty} t_n(e^{-2n\pi ix}-1)$$

$$+\sum_{\alpha}^{\sigma=0} A_\alpha\{\operatorname{ctn}\pi\alpha+\operatorname{ctn}\pi(x-\alpha)\}.$$

Henceforth let $a=1$, $b=7$. The last sum in (20) is independent of k and must be equal to $7^{1/2}$ in view of (1). Since the denominators p of 2α are 1, 2, or 3 the calculation of s_n, c_n, t_n may be done conveniently modulo 6. For positive integers r, we find that

$$t_{6r} = (-1)^r q^{6r} + q^{3r} + q^{9r} - q^{2r} - q^{4r} - q^{8r} - q^{10r},$$

$$t_{6r-5} = q^{6r-5}, \quad t_{6r-4} = q^{3r-2} + \{1+(-1)^r\}q^{6r-4} + q^{9r-6},$$

$$t_{6r-3} = -q^{2r-1} + q^{4r-2} + q^{8r-4} - q^{10r-5},$$

$$t_{6r-2} = q^{3r-1} + \{1+(-1)^{r+1}\}q^{6r-2} + q^{9r-3}, \quad t_{6r-1} = q^{6r-1}.$$

And for any integer n,

$$c_n = \tfrac{1}{2}\{t_n + t_{-n}\} + \gamma_n,$$

where γ_n is the sum of the terms $Ae^{2n\rho i\pi}$ for $\sigma=0$, that is,

$$\gamma_{6r} = (-1)^r, \gamma_{6r-5} = -1, \qquad \gamma_{6r-4} = 1+(-1)^r,$$

$$\gamma_{6r-3} = 0, \qquad \gamma_{6r-2} = 1+(-1)^{r+1}, \gamma_{6r-1} = -1.$$

Since t_n is real for every n, and since

$$s_n = -\tfrac{1}{2}i(t_n - t_{-n}),$$

s_n is purely imaginary. We may drop the imaginary parts, which must annul each other. Hence

$$(21)\quad 7^{1/2}\sum_{x,y} q^{x^2+7y^2} = 7^{1/2} + 2\sum_{n=1}^{\infty} t_n \sin\frac{2n\pi}{7} + 4\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\, c_n \sin\frac{2n\pi}{7}.$$

Inserting the values of t_n, c_n calculated above, we find for the right member of (21) the expression

$$(22)\quad \begin{aligned} &7^{1/2} + 4\sum\frac{q^{6r-5}}{1+q^{6r-5}}\sin(12r-10)\frac{\pi}{7} \\ &+ 4\sum\frac{q^{6r-1}}{1+q^{6r-1}}\sin(12r-2)\frac{\pi}{7} \\ &+ 4\sum\left(\frac{q^{3r-2}}{1-q^{3r-2}} + \frac{(-1)^r q^{6r-4}}{1-q^{6r-4}}\right)\sin(12r-8)\frac{\pi}{7} \\ &+ 4\sum\left(\frac{q^{3r-1}}{1-q^{3r-1}} + \frac{(-1)^{r+1} q^{6r-2}}{1-q^{6r-2}}\right)\sin(12r-4)\frac{\pi}{7} \\ &+ 4\sum\frac{q^{4r-2}-q^{2r-1}}{1+q^{6r-3}}\sin(12r-6)\frac{\pi}{7} \\ &+ 4\sum\frac{(-1)^r q^{6r}+q^{3r}-q^{2r}-q^{4r}}{1-q^{6r}}\sin\frac{12r\pi}{7}. \end{aligned}$$

The coefficient of q^n in (22) may be calculated as follows. Write $f(m) = 2\sin(2m\pi/7)$. Form all factorizations $n = d\delta$ of n into two positive factors and construct the following sums:

$$\sum_1 f(d)(-1)^{n+1}, \quad d \equiv 1, 5 \pmod 6 \qquad \sum_2 f(2d), \quad d \equiv 1, 2 \pmod 3$$

$$\sum_3 f(d)(-1)^{d/2}, \quad d \equiv 0, 2, 4 \pmod 6 \qquad \sum_4 f(3d)(-1)^n, \quad d \equiv 1 \pmod 2,\ \delta \equiv 1, 2 \pmod 3$$

$$\sum_5 f(2d), \quad d \equiv 0 \pmod 3,\ \delta \equiv 1 \pmod 2 \qquad \sum_6 [-f(3d)]. \quad d \equiv 0 \pmod 2,\ \delta \equiv 1, 2 \pmod 3$$

Hence we have

$$(23)\quad 7^{1/2}N(n = x^2 + 7y^2) = 2\left(\sum_1 + \sum_2 + \cdots + \sum_6\right).$$

If $n=m$ is prime to 6, this reduces to

$$7^{1/2}N(m = x^2 + 7y^2) = 2\sum_{d|m}[f(d) + f(2d) - f(3d)]. \tag{24}$$

We now make use of a (Gauss sum) relation, which may be proved if desired by comparing coefficients of a small power of q in (21):

$$\sin\frac{2d\pi}{7} + \sin\frac{4d\pi}{7} - \sin\frac{6d\pi}{7} = \frac{1}{2}\left(\frac{d}{7}\right)7^{1/2}. \tag{25}$$

Hence, if m is prime to 6,

$$N(m = x^2 + 7y^2) = 2\sum(d \mid 7). \tag{26}$$

Making use of (23) for one case we can easily establish by induction the formula

$$N(n = x^2 + 7y^2) = 2\,|\,\alpha - 1\,|\ \sum(d \mid 7), \tag{27}$$

where $n=2^\alpha m$, m odd, $\alpha \geqq 0$.

The factor $|\alpha-1|$ for a primary component 2^α is characteristic of a discriminant $d=4D$, $D\equiv 1 \pmod 8$.

CALIFORNIA INSTITUTE OF TECHNOLOGY

QUADRATIC PARTITIONS—I

BY E. T. BELL

1. *Introduction.* This is the preliminary and longest note of a series which, by the kindness of the editors of this Bulletin, I hope to publish from time to time, giving some of the numerous general arithmetical theorems of a particular type which I have been accumulating for several years. To make this series self contained, I first recall the necessary definitions, and give once for all a few formulas that will be used repeatedly. Results and methods of two previous papers are indicated by numbered references.*

Subsequent notes will contain only theorems, with statements of the elementary identities from which they follow. This will be sufficient to enable anyone who wishes to retrace the details of the proofs and verify the conclusions. I believe that present conditions of mathematical publication in this country demand the utmost brevity consistent with reasonable clarity.

2. *Parity.* Let $\xi=(x_1, \cdots, x_n)$ be a one-row matrix or vector in which the *elements* $x_1, \cdots, x_n$ are in a given field K. Write $-\xi\equiv(-x_1, \cdots, -x_n)$. If $f(\xi)$ is a single finite real or complex number whenever $x_1, \cdots, x_n$ are in K, we say that $f(\xi)$ is *uniform over* K. Let $f(\xi)$ be uniform over K. Then, if $f(-\xi)=f(\xi)$, we say that $f(\xi)$ has *parity* $p(n\,|\,)$ in ξ; if $f(-\xi)=-f(\xi)$, and if further $f(0, \cdots, 0)=0$, the parity is $p(\,|\,n)$. Let us denote by ξ_i, η_j, $(i=1, \cdots, r; j=1, \cdots, s)$, vectors in K, having no element in common. Then, if $f(\xi_1, \cdots, \xi_r, \eta_1, \cdots, \eta_s)$ has parity $p(n_i'\,|\,)$ in ξ_i, and parity $p(\,|\,n_j'')$ in η_j $(i=1, \cdots, r; j=1, \cdots, s)$, we shall agree to say that $f(\xi_1, \cdots, \xi_r, \eta_1, \cdots, \eta_s)$ has parity $p(n_1', \cdots, n_r'\,|\,n_1'', \cdots, n_s'')$ in $(\xi_1, \cdots, \xi_r\,|\,\eta_1, \cdots, \eta_s)$, and we write

* (1) *Arithmetical paraphrases*, Transactions of this Society, vol. 22 (1921), pp. 1–30; 198–219; (2) *A revision of the Bernoullian and Eulerian functions*, this Bulletin, vol. 28 (1922), pp. 443–450. The material in (1) is included and generalized in (3) *Algebraic Arithmetic*, American Mathematical Society Colloquium Publications, vol. 7, 1927, Chapters 2, 3; ibid., pp. 146–159, contain a complete account of the umbral calculus used in (2) and in some of the present notes.

$$f(\xi_1, \cdots, \xi_r, \eta_1, \cdots, \eta_s) \equiv f(\xi_1, \cdots, \xi_r \mid \eta_1, \cdots, \eta_s).$$

The field K will be given explicitly, or by the context. Unless otherwise noted, K is the field of all rational numbers, and the values of the elements of $\xi_1, \cdots, \xi_r, \eta_1, \cdots, \eta_s$ are rational integers.

3. *Notation.* As in the references in §1, $n, n_i, d, \delta, d_i, \delta_i, t, t_i, m, \tau, m_i, \tau_i, \nu, \nu_i, a, b, \mu, \mu_i, (i=1, \cdots)$, denote integers, of which $n, n_i, d, \delta, d_i, \delta_i, t, t_i$ are greater than zero, and otherwise unrestricted, m, τ, m_i, τ_i are greater than zero and odd, ν, ν_i, a, b are greater than, equal to, or less than zero and are unrestricted, μ, μ_i are greater than or less than zero and odd.

If one or more of $n, \cdots, \mu_i$ occur under $\sum$, the sum refers to all $n, \cdots, \mu_i$ as defined.

A sum $\sum_a^b$ in which $b<a$ is vacuous, and is to be suppressed.

The *umbra* (see §1, references (2), (3)) of the sequence $\xi_0, \xi_1, \cdots, \xi_s, \cdots$, in which the first element has the suffix zero, is ξ. Symbolically, $\xi^s \equiv \xi_s, (s=0, 1, \cdots)$. I define the (umbral) *indefinite integral* of ξ to be ξ', where ξ' is the umbra of $\xi_{s+1}/(s+1), (s=0, 1, \cdots)$. The even suffix notation is used (as in paper (2)) for the numbers of Bernoulli, Euler, Genocchi, and Lucas, whose respective umbrae are B, E, G, R. Hence B', E', G', R' are defined. The sequences of functions associated with B, G, E, R, whose respective umbrae are $\beta, \gamma, \eta, \rho$, are as in the paper (2). If necessary to indicate the argument x, we shall write $\beta(x)$, etc. Thus $\beta(x)$ is the umbra of $\beta_s(x), (s=0, 1, \cdots)$; $\beta'(x)$ is the umbra of $\beta_{s+1}(x)/(s+1), (s=0, 1, \cdots)$.

4. *Partitions.* Let n (§3) be constant, and let $Q(x_1, \cdots, x_p)$ be any polynomial in $x_1, \cdots, x_p$ with coefficients in K(§2). The totality of vectors $(x_1, \cdots, x_p)$, whose elements are in K, such that $n=Q(x_1, \cdots, x_p)$, will be called the *Q-partition* of n. If this partition contains an infinity of distinct vectors, we impose conditions $C(x_1, \cdots, x_p)$ upon $x_1, \cdots, x_p$ such that, subject to $C(x_1, \cdots, x_p)$, the Q-partition contains only a finite number of distinct vectors, and refer to this as a *restricted* partition. Restrictions will always be stated explicitly; otherwise, the partition is unrestricted. If $x_1, \cdots, x_p$ occur under $\sum$, the sum is with respect to the partition, and the limits need not be otherwise indicated.

If Q above is homogeneous of degree 2, the partition is called *quadratic*.

5. *Special Functions.* The following will occur frequently. If x is real and positive, $[x]$ in an exponent or as a summation limit denotes the greatest integer in x. If y is real and different from zero, sgn y is defined (as usual) by sgn $y = y^{-1}|y|$, and sgn $0 = 0$. Hence, for real u, v,

$$\sin(u \operatorname{sgn} v) = \operatorname{sgn} v \sin u;$$

for z real, $\neq 0$, and x, y real,

$$\cos(x \operatorname{sgn} z) = \cos x,$$

$$\operatorname{sgn} z \cos x \cos y \pm \sin x \sin y = \operatorname{sgn} z \cos(x \mp y \operatorname{sgn} z),$$

$$\operatorname{sgn} z \sin x \cos y \pm \cos x \sin y = \operatorname{sgn} z \sin(x \pm y \operatorname{sgn} z).$$

Referring to §3, we define $\epsilon(\nu)$ to be $+1$ if ν is even; -1 if ν is odd. Refer to §2 for ξ. If $f(\xi)$ has an expansion of the form

$$\sum A_{a_1,\cdots,a_n} x_1^{a_1} \cdots x_n^{a_n},$$

which is convergent in some non-zero region of the n-space of ξ, we say that $f(\xi)$ is an *entire function of* ξ. In particular, a polynomial in $x_1, \cdots, x_n$ is entire in ξ.

The previous notation (paper (1), p. 207) $\phi_{abc}(x, y)$ for the doubly periodic functions of the second kind,

$$\phi_{abc}(x, y, q) \equiv \phi_{abc}(x, y) = \vartheta_1' \vartheta_a(x + y)/(\vartheta_b(x)\vartheta_c(y)),$$

of which there are 16, will be used. The remaining 48 expansions, not available in previous work, have been obtained by D. A. F. Robinson, and will be printed elsewhere.*

6. *Special Umbral Identities.* In the passage from trigonometric identities to their equivalents in terms of parity functions, the trigonometric terms having simple poles at the origin play a particular part; see paper (1), p. 204. Such terms contribute sums of parity functions one or more of whose arguments are in arithmetical progression. The residue of the pole must be zero in any trigonometric identity paraphrased. If it is not immediately obvious that the residue vanishes, the fact that it must gives a subsidiary theorem. The following formulas, which will be frequently used, enable us to write down the residues without calculations in one type of theorem; ξ is umbral as in §3.

* Probably in the Transactions of the Royal Society of Canada. These expansions will be stated when used.

$$2 \operatorname{ctn} x \sin(\xi x + y) = 2x^{-1}\xi_0 \sin y + \cos\{\beta'(\xi)x + y\},$$
$$2 \operatorname{ctn} x \cos(\xi x + y) = 2x^{-1}\xi_0 \cos y - \sin\{\beta'(\xi)x + y\};$$

$$4 \tan x \sin(\xi x + y) = \cos\{\gamma'(\xi)x + y\},$$
$$4 \tan x \cos(\xi x + y) = -\sin\{\gamma'(\xi)x + y\};$$

$$2 \sec x \sin(\xi x + y) = \sin\{\eta(\xi)x + y\},$$
$$2 \sec x \cos(\xi x + y) = \cos\{\eta(\xi)x + y\};$$

$$\csc x \sin(\xi x + y) = x^{-1}\xi_0 \sin y + \cos\{\rho'(\xi)x + y\},$$
$$\csc x \cos(\xi x + y) = x^{-1}\xi_0 \cos y - \sin\{\rho'(\xi)x + y\}.$$

7. *Trigonometric Identities.* From the identities on pages 204–5 of paper (1), we write down eight which generalize them and greatly reduce algebraic work later. Refer to §§3, 5, and write

$$N \equiv [\tfrac{1}{2}|\nu|], \quad M \equiv [\tfrac{1}{2}(|\nu| - 1)].$$

Then

$$\csc x \sin(\nu x + y) = [\{1 - e(\nu)\} \operatorname{ctn} x + e(\nu) \csc x] \sin y$$
$$+ \operatorname{sgn}\nu[\{1 - e(\nu)\} \cos y + 2\sum_{r=1}^{N} \cos\{(2r - e(\nu))x \operatorname{sgn}\nu + y\}];$$

$$(-1)^N \sec x \sin(\nu x + y) = e(\nu) \sec x \sin y$$
$$+ \{1 - e(\nu)\} \operatorname{sgn}\nu \tan x \cos y + \{1 - e(\nu)\} \sin y$$
$$+ 2\sum_{r=1}^{N}(-1)^r \sin\{(2r - e(\nu))x \operatorname{sgn}\nu + y\};$$

$$\tan x \sin(\nu x + y) = (-1)^N e(\nu) \tan x \sin y$$
$$+ (-1)^M\{1 - e(\nu)\} \operatorname{sgn}\nu \sec x \cos y(-1)^M$$
$$+ (-1)^M \operatorname{sgn}\nu[e(\nu) \cos y - (-1)^M \cos(\nu x + y)$$
$$+ 2\sum_{r=1}^{M}(-1)^r \cos\{(2r - 1 + e(\nu))x \operatorname{sgn}\nu + y\}];$$

$$\operatorname{ctn} x \sin(\nu x + y) = [e(\nu) \operatorname{ctn} x + \{1 - e(\nu)\} \csc x] \sin y + \operatorname{sgn} \nu \Big[e(\nu) \cos y + \cos(\nu x + y) + 2 \sum_{r=1}^{M} \cos\{(2r - 1 + e(\nu))x \operatorname{sgn} \nu + y\} \Big].$$

The remaining four are written down from these by replacing y by $y+\pi/2$. All will be used to reduce terms involving csc, sec, tan, ctn in trigonometric identities before passing to parity functions.

8. *General Umbral Identities.* The principle of paraphrase stated in paper (1), pages 4, 5, can be extended to umbral sines and cosines, identities between which paraphrase into identities between *entire* functions as defined in §5. That is, the *elements* of the one-row matrices, or vectors, in the principle as previously stated, can be replaced by umbrae. It is necessary only to define parity for functions of umbrae, and it will be sufficient to state the definitions for functions of one umbra ξ. If $f(x) \equiv f(x\,|)$ is an entire function of the ordinary x, we say that $f(\xi)$ $(\equiv f(\xi\,|))$ has *parity* $p(1\,|)$ in ξ. According to this definition and what precedes, $f(\xi)$ is of the form

$$p_0\xi_0 + p_2\xi_2 + \cdots + p_{2s}\xi_{2s} + \cdots,$$

where the series either converges or terminates. If $g(-x) = -g(x)$, we say that $g(\xi)$ $(\equiv g(\,|\xi))$ has parity $p(\,|1)$ in ξ, and $g(\xi)$ is of the form

$$p_1\xi_1 + p_3\xi_3 + \cdots + p_{2s+1}\xi_{2s+1} + \cdots.$$

The condition $g(0) = 0$ is not imposed, as it is not required here.

To see how the principle goes over to umbrae the following case will suffice. Let

$$f(\xi) = p_0\xi_0 + p_2\xi_2 + \cdots + p_{2s}\xi_{2s};$$

let x be an ordinary umbra, and $a, b, \cdots, c$ umbrae such that

$$\cos ax + \cos bx + \cdots + \cos cx \equiv 0$$

is an identity in x. Then

$$f(a) + f(b) + \cdots + f(c) = 0.$$

For, the given identity implies

$$a_{2r} + b_{2r} + \cdots + c_{2r} = 0, \quad (r = 0, 1, \cdots);$$

and therefore

$$p_0(a_0 + b_0 + \cdots + c_0) + p_2(a_2 + b_2 + \cdots + c_2) + \cdots + p_{2s}(a_{2s} + b_{2s} + \cdots + c_{2s}) = 0;$$

which is the stated conclusion.

CALIFORNIA INSTITUTE OF TECHNOLOGY

ON SYMMETRIC PRODUCTS OF TOPOLOGICAL SPACES*

BY KAROL BORSUK AND STANISLAW ULAM

1. *Introduction.* This paper is devoted to an operation that is defined for an arbitrary topological† space E and is analogous to the operation of constructing the combinatorial product spaces.‡ We shall be concerned with the topological properties of point sets defined by means of the above operation when executed on the segment $0 \leqq x \leqq 1$.

Let E be an arbitrary topological space. Let E^n denote the nth topological product of the space E, that is, the space whose elements are ordered systems $(x_1, x_2, \cdots, x_n)$ of points $x_i \epsilon E$. By a *neighborhood* of a point $(x_1, x_2, \cdots, x_n)$, we understand the set of all systems $(x_1', x_2', \cdots, x_n')$, where x_i' belongs to a neighborhood u_i of the point x_i in the space E.‡

The operation with which we are concerned in this paper consists in constructing a space which we shall call the nth *symmetric product* of the space E and denote by $E(n)$. Its elements are *non-ordered* systems of n points (which may be different or not) belonging to E. Two systems differing only by the order or multiplicity of elements are considered identical. A non-ordered system or simply a *set* consisting of n points $x_1, \cdots, x_n$ from the space E will be denoted by $\{x_1, x_2, \cdots, x_n\}$. If u_i is a neighborhood of the point x_i in the space E, then the set of all systems

* The definition of *symmetric products* is given below.

† In the sense of Hausdorff, *Grundzüge der Mengenlehre*, p. 228.

‡ See, for example, F. Hausdorff, *Grundzüge der Mengenlehre*, p. 102.

$\{x_1', x_2', \cdots, x_n'\}$ such that $\{x_1', x_2', \cdots, x_n'\} \subset \sum_{i=1}^{n} u_i$ and $u_i \cdot \{x_1', x_2', \cdots, x_n'\} \neq 0$ for $i = 1, 2, \cdots, n$ will be, by definition, considered as the neighborhood of the point $\{x_1, x_2, \cdots, x_n\}$ in the space $E(n)$.

It may be important to observe that the sets $E(n)$ constitute a monotonic sequence of subsets of the space 2^E,* that is, the space whose elements are compact subsets of the space E.† The sets $E(n)$, with finite dimensionality,‡ if E is compact and of finite dimensions, approximate in a certain sense the space 2^E; we have, in fact, the following formula:

$$2^E = \overline{\sum_{n=1}^{\infty} E(n)}.$$

The study of $E(n)$ may, therefore, throw some light on the structure of 2^E.§

In the case where E is a metric space and $|x' - x|$ denotes the distance of two points x, $x' \epsilon E$, we may consider E^n, and $E(n)$ also, as metric and by means of the following formulas define the distance of two points of these spaces:

$$|(x_1, x_2, \cdots, x_n) - (x_1', x_2', \cdots, x_n')| = \left(\sum_{i=1}^{n} |x_i - x_i'|^2\right)^{1/2}$$

and

$$|\{x_1, x_2, \cdots, x_n\} - \{x_1', x_2', \cdots, x_n'\}| = \operatorname{Max}\left[\sup_{1 \leq i \leq n} \inf_{1 \leq j \leq n} |x_i - x_j'|;\ \sup_{1 \leq i \leq n} \inf_{1 \leq j \leq n} |x_j - x_i|\right],$$

respectively.¶

We shall show that, generally speaking, the operations of constructing the combinatorial and the symmetric product of the space E lead to topologically different results. Thus in the

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* This very convenient notation was introduced by C. Kuratowski, Fundamenta Mathematicae, vol. 17.

† F. Hausdorff, *Grundzüge der Mengenlehre*, p. 293.

‡ See (b), §2.

§ The space 2^E has been recently an object of some studies. See F. Hausdorff, *Grundzüge der Mengenlehre*, p. 145; S. Mazurkiewicz, Fundamenta Mathematicae, vol. 16, p. 151, and others.

¶ See Hausdorff, loc. cit.

case where E is a one-sphere, that is, the circumference of a circle, E^2 is the surface of an anchor ring (torus) and $E(2)$ is the well known one-sided Möbius strip. Neither of these sets is topologically contained in the other. This example may suffice to show that the symmetric products are apt to be used as means of simple definitions of interesting topological spaces.

In this paper, however, we shall be concerned with the space $I(n)$, I being the segment $0 \leqq x \leqq 1$. The first question to be treated is whether or not $I(n)$ is topologically equivalent with a subset of the n-dimensional euclidian space R^n (R is the set of all real numbers).

We formulate our problem in the following manner in order to emphasize certain algebraic analogies.

A real-valued function $\phi(x_1, x_2, \cdots, x_n)$ of n real variables will be called essentially symmetric if and only if its value depends upon the set $x_1, x_2, \cdots, x_n$ and not upon the order or multiplicity of the x's. Our problem may now be given the following formulation: Does there exist a system of functions $\phi_i(x_1, x_2, \cdots, x_n)$, $(i = 1, 2, \cdots, n)$, which satisfies the conditions (a) the functions ϕ_i are essentially symmetric and continuous for $0 \leqq x \leqq 1$; (b) the system of equations $\phi_i(x_1, x_2, \cdots, x_n) = y_i$ has at most one solution for every system $y_1, y_2, \cdots, y_n$?

We show in §9 that the answer to our question is affirmative for $i = 1, 2, 3$ and negative for $i \geqq 4$.

2. *Invariants.* Let us consider a function ϕ defined on the set E^n by the formula

$$\phi(x_1, x_2, \cdots, x_n) = \{x_1, x_2, \cdots, x_n\}.$$

This function transforms E^n on $E(n)$ continuously and every point belonging to $E(n)$ is an image of at most $n!$ points of the set E^n. If we recall well known properties* of the space E^n and continuous transformations, we obtain the propositions:

(a) *The properties local-connectedness, separability, compactness, arcwise connectedness, absolute G_δ, compactness and local connectedness at once, are invariants under the operation of constructing the symmetric product.*

(b) *If E is compact, then*

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* F. G. van Dantzig, Fundamenta Mathematicae, vol. 15, p. 117.

$\operatorname{Dim} E^n \leqq \operatorname{Dim}(E(n)) \leqq \operatorname{Dim} E^n + n! - 1 \leqq n^* \cdot \operatorname{Dim} E + n! - 1$.†

The question whether or not the following topological properties are invariants by the operation of symmetric product remains unsolved:

(α) *To be a locally connected unicoherent continuum.*

(β) *To possess a fixed point.*

(γ) *To be an n-dimensional Cantor manifold.*

(δ) *To be an absolute retract.*‡

It may also be interesting to know the exact relation between the dimensions of E and $E(n)$. Finally we wish to find the relation, if any, between the combinatorial characters, that is, the Betti and torsion numbers, of E and $E(n)$.

3. THEOREM 1. *Let A be an everywhere dense subset of the dense space E. Then $A(n) - E(n-1) = E(n)$ for $n = 2, 3, \cdots$.*

Let $\{x_1, x_2, \cdots, x_n\} \epsilon E(n)$ and U_i be, for $i = 1, 2, \cdots, n$, any arbitrary neighborhood of the point x_i in E.

We have to show that there exists a set $\{x_1', x_2', \cdots, x_n'\} \epsilon A(n) - E(n-1)$ such that $\{x_1', x_2', \cdots, x_n'\} \subset \sum_{i=1}^{n} U_i$ and

$$\{x_1', x_2', \cdots, x_n'\} \cdot U_i \neq 0 \text{ for } i = 1, 2, \cdots, n.$$

The set A constitutes an everywhere dense subset of E. Hence for each $i = 1, \cdots, n$, there exists a sequence $[x_n^{(i)}]$ consisting of different points of the set $A \cdot U_i$. Putting $x_1' = x_1^{(1)}$ and supposing that for a certain p such that $1 \leqq p \leqq n$ the points $x_2' = x_{k_p}^{2)}, \cdots, x_p' = x_{k_p}^{(p)}$ are already defined, we put $x_{p+1}^{1)}$ equal to the first term from the sequence $x_k^{(p+1)}$ different from all x_i'. Thus we obtain n different points $x_1', x_2', \cdots, x_n' \epsilon A$ or a point $\{x_1', x_2', \cdots, x_n'\}$ from the set $A(n) - E(n-1)$, such that $x_p' = x_{k_p}^{(p)} \epsilon U_p$. Hence the theorem is proved.

4. THEOREM 2. *The set $I(n) - I(n-1)$, for $n = 2, 3, \cdots$, is homeomorphic with a subset of R^n.*

Let T be a subset of R^n consisting of all points $(x_1, x_2, \cdots, x_n)$ which fulfill the conditions $0 \leqq x_1 < x_2 < \cdots < x_n \leqq 1$. For

* W. Hurewicz, Proceedings of the Amsterdam Academy, vol. 30, p. 164.

† K. Menger, *Dimensionstheorie*, p. 246.

‡ A subset B of A is a *retract* of A if there exists a continuous function f, defined on A, such that $f(A) = B$ and for every $x \epsilon B$, $f(x) = x$. An *absolute retract*, is, by definition, a homeomorph of a retract of the fundamental Hilbert cube. See K. Borsuk, Fundamenta Mathematicae, vol. 17, p. 153 and p. 159.

$(x_1, x_2, \cdots, x_n) \epsilon T$ let us put $\psi(x_1, x_2, \cdots, x_n) = \{x_1, x_2, \cdots, x_n\}$. It is evident that ψ is a homeomorphism which carries T on $I(n) - I(n-1)$, which was to be proved.

5. THEOREM 3. Dim $I(n) = n$.

In view of 1 (b) and the known fact that Dim $I^n = n$ it will be sufficient to prove that

$$\text{Dim } I(n) \leqq n. \tag{1}$$

The truth of inequality (1) is evident if $n = 1$, that is, $I(1)$ is identical with the segment $0 \leqq x \leqq 1$. Let us suppose now that for a certain k the inequality (1) is proved. The set $I(k+1) - I(k)$ is open ((a), §2); hence it is an F_σ of dimension at most $k+1$.

Regarding equality $I(k+1) = I(k) + [I(k+1) - I(k)]$ and applying Menger's "Summensatz,"* we obtain the desired inequality.

6. THEOREM 4. *Let A denote the segment I without its two ends: $A = I - (0) - (1)$. Then, for each pair of points, $\{x_1', x_2', \cdots, x_n'\}$ and $\{x_1'', x_2'', \cdots, x_n''\}$ of set $A(n) - I(n-1)$ for $n = 2, 3, \cdots$, there exists a subset of $A(n) - I(n-1)$ homeomorphic with I^n and containing both points.*

PROOF. We do not diminish the generality if we put

$$\begin{cases} 0 < x_1' < x_2' < \cdots < x_n' < 1, \\ 0 < x_1'' < x_2'' < \cdots < x_n'' < 1, \end{cases} \tag{2}$$

and for a certain i_0, $(1 \leqq i_0 \leqq n)$,

$$x_{i_0}' < x_{i_0}''. \tag{3}$$

Let us put for each $(t_1, t_2, \cdots, t_n) \epsilon R^n$

$$\begin{cases} x_i(t_1, t_2, \cdots, t_n) = x_i' + t_{i_0}(x_i'' - x_i') + t_i, \text{ for } i \neq i_0, \\ x_{i_0}(t_1, t_2, \cdots, t_n) = x_{i_0} + t_{i_0}(x_{i_0}'' - x_{i_0}'). \end{cases} \tag{4}$$

From (2) and (4) it follows that there exists a positive number α such that the inequalities

$$0 \leqq t_{i_0} \leqq 1, \text{ and } |t_i| \leqq \alpha \tag{5}$$

include the inequalities

* See K. Menger, *Dimensionstheorie*, p. 93.

$$(6)\quad 0 < x_1(t_1, t_2, \cdots, t_n) < x_2(t_1, t_2, \cdots, t_n) < \cdots < x_n(t_1, t_2, \cdots, t_n) < 1.$$

The points of R^n with the coordinates $t_1, t_2, \cdots, t_n$ which satisfy the inequality (5) form in R^n a set P_n homeomorphic to I^n. For each point $(t_1, t_2, \cdots, t_n) \epsilon P_n$ let us put

$$f(t_1, t_2, \cdots, t_n) = \{x_1(t_1, t_2, \cdots, t_n), x_2(t_1, t_2, \cdots, t_n), \cdots, x_n(t_1, t_2, \cdots, t_n)\}.$$

This is a continuous function on the compact set P_n and, with regard to (3) and (4), a one-to-one correspondence (the second of the equations (4) can be solved for t_{i_0}, the rest for t_i, $i \neq i_0$). Since the function f carries P_n homeomorphically on a subset of the set $A(n) - I(n-1)$ (from (6)) and

$$f(0, 0, \cdots, 0) = \{x_1', x_2', \cdots, x_n'\},$$
$$f(0, 0, \cdots, 0, 1, 0, \cdots, 0) = \{x_1'', x_2'', \cdots, x_n''\},$$

our theorem is proved.

7. THEOREM 5. *$I(n)$ is an n-dimensional, locally-connected Cantor manifold.**

In view of (a), §1, and Theorem 3, it remains to be shown that no compact $(n-2)$-dimensional subset C from $I(n)$ cuts $I(n)$. Since I^n is an n-dimensional Cantor-manifold,† it follows from (5) that $A(n) - I(n-1) - C$ is connected and everywhere dense in $A(n) - I(n-1)$. Hence, it follows that from §3

$$A(n) - I(n-1) - C \subset I(n) - C \subset I(n) = \overline{A(n) - I(n-1)} = \overline{A(n) - I(n-1) - C},$$

which includes the fact that $I(n) - C$ is connected.

8. THEOREM 6. *For $n = 1, 2, 3$, $I(n)$ is a homeomorph of I^n.*

PROOF. Let $\{x, y, z\} \epsilon I(3)$. Since order and multiplicity do not matter, we can suppose that $0 \leqq x \leqq y \leqq z \leqq 1$. Moreover, $x = y$ if and only if $x = y = z$. Let us put

$$(7)\qquad f(\{x, y, z\}) = (\xi(x, y, z), \eta(x, y, z), \zeta(x, y, z))$$

* See, for example, K. Menger, *Dimensionstheorie*, p. 217.
† K. Menger, *Dimensionstheorie*, p. 268.

where the coordinates ξ, η, ζ of the points $(\xi, \eta, \zeta) \epsilon R^3$ are defined by the formulas

$$(8) \qquad \xi(x, y, z) = \begin{cases} (z - x) \cdot \sin\left(2\pi \cdot \dfrac{y - z}{x - z}\right), & \text{if } z > x; \\ 0, & \text{if } z = x; \end{cases}$$

$$(9) \qquad \eta(x, y, z) = \begin{cases} (z - x) \cdot \cos\left(2\pi \cdot \dfrac{y - z}{x - z}\right), & \text{if } z > x; \\ 0, & \text{if } z > x; \end{cases}$$

$$(10) \qquad \zeta(x, y, z) = x.$$

It is easy to observe that f transforms $I(3)$ homeomorphically on a cone S, whose base is a circle in the plane $\zeta = 0$ with the center (0, 0, 0) and radius 1 and whose vertex is the point (0, 0, 1); $f(I(2))$ is a triangle with vertices (0, 0, 0), (0, 1, 0) and (0, 0, 1). Finally $f(I(1))$ is a segment L with the ends (0, 0, 0) and (0, 0, 1). As a cone is homeomorphic with I^3 and a triangle with I^2, we obtain our theorem.

9. THEOREM 7. *For $n \geqq 4$, $I(n)$ is not homeomorphic with any subset of R^n.*

Let J denote the segment $-1 \leqq x \leqq 1$. Noting that $I(n)$ and $J(n)$ are homeomorphic, it will be sufficient to prove that there does not exist a homeomorphism h between $J(n)$ and a subset of R^n. Let

$$(11) \quad Q = E_{\{x, y, z, x_1, \cdots, x_{n-3}\}} \left[0 \leqq y \leqq z \leqq 1, y = x \text{ if and only if } x = y = z, \right.$$

$$\left. \text{and } \left| x_i + \frac{i}{n} \right| \leqq \frac{1}{3n} \text{ for } 1 \leqq i \leqq n - 3\right].$$

Let us say, for each $\{x, y, z, x_1, x_2, \cdots, x_{n-3}\} \epsilon Q_3$, where $x, y, z, x_1, x_2, \cdots, x_{n-3}$ are chosen in such a way that the inequalities (11) are satisfied, that

$$\phi(\{x, y, z, x_1, x_2, \cdots, x_{n-3}\}) = (\xi, \eta, \zeta, \xi_1, \xi_2, \cdots, \xi_{n-3}),$$

where ξ, η, ζ are defined by (8), (9), (10), and

$$(12) \qquad \xi_i = x_i + \frac{i}{n}, \qquad (i = 1, 2, \cdots, n - 3).$$

From the definition of ϕ and the properties of f expressed by

(7) it follows immediately that $\phi(Q)$ is a combinatorial product* of S and an $(n-3)$-dimensional product of the segment $[-1/(3n), 1/(3n)]$, that is, topologically a homeomorph of J^n. The point $f(\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\})$ is (by (7), (8), (9) and (10)) an inner point of the cone S; it follows that the point

$$\phi\left(\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{n}, -\frac{2}{n}, \cdots, \frac{n-3}{n}\right\}\right) \tag{13}$$

is an inner point of $\phi(Q)$.

Applying Brouwer's theorem† of the invariance of region in R^n, we conclude that the point $h(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1/n, -2/n, \cdots, -(n-3)/n)$ is an inner point of $h(J(n))$. Now let us consider the sequence $[p_k]$ of points from the set $J(n)$:

$$p_k = \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{n} + \frac{1}{3kn}, -\frac{1}{n}, -\frac{2}{n}, \cdots, -\frac{n-3}{n}\right\}.$$

We have p_k *non* ϵQ and

$$\lim_{k\to\infty} p_k = \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{n}, -\frac{2}{n}, \cdots, -\frac{n-3}{n}\right\}.$$

Noting that h is a homeomorphism, we have further $h(p_k)\epsilon R^n - h(Q)$ and

$$\lim_{k\to\infty} h(p_k) = h\left(\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{n}, -\frac{2}{n}, \cdots, -\frac{n-3}{n}\right\}\right),$$

which is impossible by (13). This proves our theorem.

It may be interesting to know whether or not $I(n)$ is homeomorphic with a subset of R^{n+1}.

LWOW, POLAND

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* See Hausdorff, *Mengenlehre*.

† L. E. J. Brouwer, Mathematische Annalen, vol. 71, pp. 305–313.

ON THE APPLICATION OF MARKOFF'S THEOREM TO PROBLEMS OF APPROXIMATION IN THE COMPLEX DOMAIN*

BY DUNHAM JACKSON

In a recent note,† the writer has discussed an extension to the complex domain of a method previously used in connection with problems of the approximate representation of real functions, in which the proof of convergence is based on Bernstein's theorem on the derivative of a polynomial or trigonometric sum. The discussion for the case of a complex variable involved certain restrictions on the boundary of the region with which the problem was concerned. The object of the present paper is to show how these restrictions may be somewhat lightened, at the expense, to be sure, of a compensating increase in the stringency of the hypotheses on the function to be approximated, if Markoff's theorem on the derivative of a polynomial is used in place of that of Bernstein.

Markoff's theorem may be stated for the purpose in hand as follows.

If $P_n(z)$ is a polynomial of the nth degree such that $|P_n(z)| \leqq L$ at all points of a line segment of length $2h$ in the z-plane, then $|P_n'(z)| \leqq n^2L/h$ at all points of the same segment.

The theorem is commonly stated for an interval of the axis of reals, and more particularly for the interval $(-1, 1)$, thus:‡ If $|P_n(x)| \leqq L$ for $-1 \leqq x \leqq 1$, then $|P_n'(x)| \leqq n^2L$ throughout the interval. If the hypothesis of the more general statement is given for the segment connecting the points z_1 and z_2, the substitution $z = \alpha + \beta z'$, with $\alpha = \frac{1}{2}(z_1 + z_2)$, $\beta = \frac{1}{2}(z_2 - z_1)$, $|\beta| = h$, reduces one to the other. In the formulation for the interval $(-1, 1)$, to be sure, the coefficients are ordinarily thought of as

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* Presented to the Society, September 8, 1931.

† *On certain problems of approximation in the complex domain*, this Bulletin, vol. 36 (1930), pp. 851–857.

‡ See, for example, Marcel Riesz, *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354–368; pp. 359–360.

real, but this restriction is not necessary.* For multiplication of the polynomial by a constant of modulus unity does not change either its absolute value or the absolute value of its derivative; and if the maximum M of the absolute value of the derivative is attained for $x = x_0$ there is no loss of generality in assuming that a suitable constant factor has been introduced to give $P_n'(x_0)$ itself the real value M. Then, if $u_n(x)$ and $iv_n(x)$ are the real and pure imaginary parts of $P_n(x)$, $u_n'(x_0) = M$ and $v_n'(x_0) = 0$, and since $|u_n(x)| \leqq |P_n(x)| \leqq L$ throughout the interval, application of the theorem to the real polynomial $u_n(x)$ gives $M \leqq n^2 L$.

Let R be any (finite) region of the plane† for which there is a positive number h such that from every point of the boundary a line segment of length $2h$ can be drawn belonging wholly to the region. Let $P_n(z)$ be a polynomial of the nth degree, and let L be an upper bound for its absolute value in R. (By the maximum-minimum theorem, together with the continuity of $P_n(z)$, the requirement in the last clause is fulfilled if $|P_n(z)| \leqq L$ throughout the interior of R, or, on the other hand, if $|P_n(z)| \leqq L$ everywhere on the boundary.) Under the conditions stated, every point of the boundary is a point of a line segment of length at least $2h$, on which $|P_n(z)|$ does not exceed L. By Markoff's theorem, consequently, $|P_n'(z)| \leqq n^2 L/h$ at every point of each of these segments, and in particular at every point of the boundary of R. From this it follows further that $|P_n'(z)| \leqq n^2 L/h$ throughout the interior of R. The result may be summarized for reference as follows.

If R is a region of the character specified above, and if $P_n(z)$ is a polynomial of the nth degree such that $|P_n(z)| \leqq L$ throughout R, then $|P_n'(z)| \leqq kn^2 L$ throughout R, k being a constant which depends only on R.

By way of commentary on the scope of the theorem, it may be noted that *the hypothesis with regard to R is satisfied if R is*

* The admissibility of complex coefficients is pointed out by Riesz, loc. cit. For a transition similar to that made here see also S. Bernstein, *Leçons sur les Propriétés Extrémales et la Meilleure Approximation des Fonctions Analytiques d'une Variable Réelle*, Paris, 1926, p. 45.

† In connection with the general problem of the extension of Markoff's theorem, as well as that of Bernstein, to the complex domain, see also G. Szegö, *Über einen Satz von A. Markoff*, Mathematische Zeitschrift, vol. 23 (1925), pp. 45–61; for the case of Bernstein's theorem on a circle in the complex plane, M. Riesz, loc. cit., p. 357, and S. Bernstein, op. cit., pp. 44–45.

any convex region bounded by a Jordan curve, or is made up of a finite number of such regions. For, in the case of a single convex region, if the boundary is divided into four segments in any way, the line joining any point of one segment to any point of the opposite segment has a positive minimum length, and belongs wholly to the region. The truth of the statement with regard to a composite region is an immediate corollary.

Now let R, whether meeting the terms of the last paragraph or not, *be a region satisfying the original hypothesis* as to the accessibility of all points of the boundary by line segments whose lengths have a positive lower bound, *and let the boundary of R be a simple closed Jordan curve*:

$$x = \phi(t), \quad y = \psi(t),$$

where $\phi(t)$ *and* $\psi(t)$ *are continuous functions with period* p, *and*

$$|\phi(t_2) - \phi(t_1)| + |\psi(t_2) - \psi(t_1)| \neq 0$$

for $0 < |t_2 - t_1| < p$. It would make no essential difference if R were a multiply connected region bounded by a finite number of such curves, or made up of a finite number of separate regions of similar character, but for simplicity the discussion will be formulated for the case of a single simply connected region.

Let $f(z)$ *be a function analytic throughout the interior of R, and continuous on the boundary. Let* $\rho(t)$ *be a bounded and measurable function of period* p, *with a positive lower bound. Let* $P_n(z)$ *be determined among all polynomials of the nth degree as the one for which the integral*

$$\int_0^p \rho(t) |f(z) - P_n(z)|^m dt, \quad z = \phi(t) + i\psi(t),$$

has the smallest possible value, m being a given positive constant. This minimum problem has a solution, and for $m > 1$ the solution is unique; the essentials of the proof, as applied to various problems of similar character, are well known.

The problem of approximation to which the above corollaries of Markoff's theorem are to be applied is that of the convergence of $P_n(z)$ toward $f(z)$ as n becomes infinite.

For this purpose *one more assumption will be made with regard to the boundary of R, namely that the functions* $\phi(t)$, $\psi(t)$ *satisfy a Lipschitz condition*:

$$|\phi(t_2) - \phi(t_1)| \leqq \lambda |t_2 - t_1|, \quad |\psi(t_2) - \psi(t_1)| \leqq \lambda |t_2 - t_1| .$$

The construction of a convergence proof then involves merely the adaptation of processes of reasoning which have been used repeatedly elsewhere.*

Let $p_n(z)$ be an arbitrary polynomial of the nth degree; let ϵ_n be the maximum of $|f(z)-p_n(z)|$ on the boundary of R; let $r_n(z)=f(z)-p_n(z)$; and let $\pi_n(z)=P_n(z)-p_n(z)$. Then

$$r_n(z) - \pi_n(z) = f(z) - P_n(z),$$

$$\gamma_n = \int_0^p \rho(t) |f(z) - P_n(z)|^m \, dt = \int_0^p \rho(t) |r_n(z) - \pi_n(z)|^m \, dt.$$

Let μ_n be the maximum of $|\pi_n(z)|$ on the boundary, z_1 a point of the boundary at which this maximum is attained, and t_1 the corresponding value of t in the interval $(0, p)$.

Markoff's theorem, as extended above, yields the fact that $|\pi_n'(z)| \leqq kn^2\mu_n$ throughout R. So

$$|\pi_n(z) - \pi_n(z_1)| = \left| \int \pi_n'(z) dz \right| \leqq \int |\pi_n'(z)| \, ds \leqq kn^2\mu_n\sigma,$$

if the integral is extended along any rectifiable path which goes from z_1 to z without passing outside of R, and if σ is the length of this path. The boundary of R is rectifiable, by virtue of the Lipschitz condition on ϕ and ψ. Hence it is readily deduced that $|\pi_n(z)|$ remains greater than $\frac{1}{2}\mu_n$ over an arc of the boundary whose length is of the order of $1/n^2$. By more specific application of the Lipschitz condition, z will remain on such an arc as t varies from t_1 over an interval having a length of the order of $1/n^2$. Reasoning of the type set forth in the various passages referred to then shows that

$$|r_n(z) - \pi_n(z)| = |f(z) - P_n(z)| \leqq cn^{2/m}\epsilon_n,$$

where c is independent of n.

For uniform convergence of $P_n(z)$ toward $f(z)$ in the interior and on the boundary of R it is sufficient that it be possible to choose polynomials $p_n(z)$ for each value of n so that $\lim_{n\to\infty} n^{2/m}\epsilon_n = 0$.

* See, for example, D. Jackson, *On the convergence of certain trigonometric and polynomial approximations*, Transactions of this Society, vol. 22 (1921), pp. 158–166; *The Theory of Approximation*, New York, 1930, pp. 82–86, 96–98; this Bulletin, loc. cit.

The restriction on $f(z)$ in this conclusion is more severe than in the earlier paper in this Bulletin, to the extent that the factor $n^{1/m}$ in that paper has been replaced by $n^{2/m}$; but the regions R admitted here are considerably more general. For example, the earlier result is not applicable to a rectangular region, which is admissible as a very special case under the present hypotheses. With respect to the question of uniform convergence throughout the given closed region, the present result is intermediate between the earlier one given by the author and those obtained by Walsh* for still more general regions, under the hypothesis that the function represented is analytic on the boundary.

The scope of the present method can be somewhat broadened, as far as the character of the region is concerned, by generalizing the original statement of Markoff's theorem in other directions.

Attention will be given next to the case of trigonometric sums in a real variable.

Let $C_n(\theta)$ be a cosine sum of the nth order, an expression of the form

$$a_0 + a_1 \cos\theta + \cdots + a_n \cos n\theta;$$

the coefficients may be real or complex. Let L be an upper bound for $|C_n(\theta)|$ in the interval $\gamma \leqq \theta \leqq \delta$, $0 < \gamma < \delta < \pi$. If $x = \cos\theta$, $C_n(\theta)$ is a polynomial of the nth degree in x, $P_n(x)$, and $|P_n(x)| \leqq L$ for $\cos\delta \leqq x \leqq \cos\gamma$. Hence $|P_n'(x)| \leqq c_1 n^2 L$ throughout the same interval, if c_1 stands for the quantity $2/(\cos\gamma - \cos\delta)$, the reciprocal of half the length of the interval over which x ranges, and

$$|C_n'(\theta)| = |P_n'(x)|\,|dx/d\theta| \leqq c_1 n^2 L.$$

(Either of the extreme values $\gamma = 0$, $\delta = \pi$ could be admitted here equally well, but they are to be ruled out in the following paragraphs. If the hypothesis is satisfied for $0 \leqq \theta \leqq \pi$, it is satisfied for all values of θ, and Bernstein's theorem is applicable.)

Let $S_n(\theta)$ be a sine sum of the nth order, and let $|S_n(\theta)| \leqq L$ for $\gamma \leqq \theta \leqq \delta$, with $0 < \gamma < \delta < \pi$ as before. It is possible to write $S_n(\theta)$ as the product of $\sin\theta$ and a cosine sum of order $n-1$, which may be denoted by $C_{n-1}(\theta)$, and the latter is a polyno-

* J. L. Walsh, *On the overconvergence of sequences of polynomials of best approximation*, Transactions of this Society, vol. 32 (1930), pp. 794–816; *Note on the overconvergence of sequences of polynomials of best approximation*, the same Transactions, vol. 33 (1931), pp. 370–388.

mial of degree $n-1$ in $x=\cos\theta$. Let this polynomial be called $P_{n-1}(x)$. Then, throughout the range specified,

$$|P_{n-1}(x)| = |C_{n-1}(\theta)| = |S_n(\theta)/\sin\theta| \leqq c_2 L,$$

if c_2 is the larger of the quantities $\csc\gamma$, $\csc\delta$. It follows that

$$\begin{aligned} |P'_{n-1}(x)| &\leqq c_1c_2(n-1)^2L \leqq c_1c_2(n^2-1)L, \\ |C'_{n-1}(\theta)| &= |P'_{n-1}(x)|\,|dx/d\theta| \leqq c_1c_2(n^2-1)L, \\ |S'_n(\theta)| &= |\sin\theta C'_{n-1}(\theta) + \cos\theta C_{n-1}(\theta)| \\ &\leqq c_1c_2(n^2-1)L + c_2L \leqq c_1c_2n^2L, \end{aligned}$$

the last inequality being justified by the fact that $c_1>1$.

Now let $T_n(\theta)$ be an arbitrary trigonometric sum of the nth order, and let $|T_n(\theta)|\leqq L$, not only for $\gamma\leqq\theta\leqq\delta$, but also for $-\delta\leqq\theta\leqq-\gamma$. In the resolution

$$T_n(\theta) = \tfrac{1}{2}[T_n(\theta) + T_n(-\theta)] + \tfrac{1}{2}[T_n(\theta) - T_n(-\theta)],$$

the expressions in brackets are a cosine sum and a sine sum respectively, and each has $2L$ as an upper bound for its absolute value in the interval $\gamma\leqq\theta\leqq\delta$. From the results of the two preceding paragraphs it is deduced that $|T'_n(\theta)|\leqq 2c_1c_2n^2L$. Since a cosine sum is an even function, and a sine sum is an odd function, this relation holds for the interval $-\delta\leqq\theta\leqq-\gamma$ also.

As the substitution $\theta'=\theta-\theta_0$ carries over a trigonometric sum of the nth order into an expression of the same form, the symmetry of the intervals $(-\delta, -\gamma)$, (γ, δ) with respect to the point $\theta=0$ has no significance; the essence of the conclusion is that if $|T_n(\theta)|\leqq L$ throughout two separate intervals of equal length, $|T'_n(\theta)|\leqq c_3n^2L$ throughout the same intervals, where c_3 is a constant depending only on the length of the intervals and the distance between them.

To pass from this to an assertion of more general utility, let $T_n(\theta)$ be an arbitrary trigonometric sum of the nth order, with real or complex coefficients, and let $|T_n(\theta)|\leqq L$ for $\alpha\leqq\theta\leqq\beta$, where α and β have any values; it may be assumed without sacrificing anything in the result that $\beta-\alpha<2\pi$, since otherwise Bernstein's theorem would be applicable. Let the interval (α, β) be divided into four equal parts. The conclusion of the last paragraph may be applied to the first and third of these parts, and again to the second and fourth parts, to show that $|T'_n(\theta)|$

$\leqq c_3 n^2 L$ throughout the whole interval (α, β), where c_3 has the appropriate value for the pairs of intervals indicated. In summary:

*If $T_n(\theta)$ is a trigonometric sum of the nth order such that $|T_n(\theta)| \leqq L$ throughout a specified interval of length less than 2π, then $|T_n'(\theta)| \leqq k' n^2 L$ throughout the same interval, k' being a constant which depends only on the length of the interval.**

To return to the complex plane, let $P_n(z)$ be a polynomial of the nth degree, and L an upper bound for $|P_n(z)|$ on an arc of the circle $|z - z_0| = R$, given by the specifications $z - z_0 = Re^{\theta i}$, $\alpha \leqq \theta \leqq \beta$. On the circumference of the circle, $P_n(z)$ reduces to a trigonometric sum of the nth order in θ, $T_n(\theta)$. For $\alpha \leqq \theta \leqq \beta$, furthermore, $|T_n(\theta)| \leqq L$. Consequently

$$|P_n'(z)| = |T_n'(\theta)|/R \leqq k' n^2 L/R.$$

Hypothesis and conclusion may be put together as follows:

If $P_n(z)$ is a polynomial of the nth degree such that $|P_n(z)| \leqq L$ at all points of a circular arc in the z-plane, then $|P_n'(z)| \leqq K n^2 L$ at all points of the same arc, where K is a constant depending only on the radius and length of the arc; for arcs of equal angular measure K may be taken inversely proportional to the radius.

It is possible now to formulate immediately a lemma of similar character, and to proceed thence to a proof of convergence of sequences of approximating polynomials, for certain two-dimensional regions to which the earlier reasoning based on Markoff's theorem for a rectilinear segment would not have been applicable; for example, a region bounded in part by two circular arcs tangent to each other internally. Space will not be taken here to give the statements at length.

The analog of Markoff's theorem can be extended directly to arcs somewhat more general than circular arcs, namely to any arc given by equations of the form $x = \phi(t)$, $y = \psi(t)$, where ϕ and ψ are polynomials or trigonometric sums such that

* For a corresponding corollary of Bernstein's theorem, see D. Jackson, *A generalized problem in weighted approximation*, Transactions of this Society, vol. 26 (1924), pp. 133–154; pp. 139–145. While there is no mention of complex coefficients in the passage cited, it is obvious that the admission of such coefficients raises a question only as to the value of the constant in the right-hand member of the inequality, and is trivial as long as the constant is left unspecified.

$|\phi'(t)|+|\psi'(t)|>0$ over the range of variation of t (inclusive of its end points). For if $P_n(z)$ is a polynomial of the nth degree it is expressible on the curve as a polynomial or trigonometric sum in t, whose degree or order is not greater than a certain constant multiple of n; and as $|dz/dt|$ is continuous and positive on the closed interval of values of t, the absolute value of the derivative of $P_n(z)$ with respect to z will not exceed a constant multiple of the absolute value of its derivative with respect to t. A simple example is an arc of the ellipse $x=a\cos t$, $y=b\sin t$.

If $\phi(t)$ and $\psi(t)$ are trigonometric sums, and if one considers the closed curve described as t ranges over a whole period, it is possible to derive similarly an analog of Bernstein's theorem, with a factor n in place of the n^2 of Markoff's theorem. The result thus obtained is much less general than that given in the author's earlier paper* with respect to the form of the functions ϕ and ψ, but possesses the advantage that the curve may have double points without introducing any additional complication.

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* This Bulletin, loc. cit., p. 856.

NOTE ON VOLTERRA AND FREDHOLM PRODUCTS OF SYMMETRIC KERNELS*

BY L. M. BLUMENTHAL

1. *Introduction.*† The purpose of this paper is to establish two theorems concerning the Volterra and the Fredholm products of two continuous, symmetric functions. These two theorems were obtained in the course of an investigation that had as its object the determination of the existence of permutable functions (first kind) of the second order; that is, the determination of functions $K_1(x, y)$, $K_2(x, y)$ such that

$$K_1K_2K_2K_1 = K_2K_1K_1K_2, \tag{1}$$

but

$$K_1K_2 \neq K_2K_1,$$

where

$$K_1K_2 = \int_x^y K_1(x, t)K_2(t, y)dt.$$

To satisfy (1),‡ it is sufficient to find two functions K_1, K_2 such that

$$K_1K_2 = -K_2K_1, \tag{2}$$

for, composing (on the right) the left-hand member with K_2K_1, and the right-hand member with its equal, $-K_1K_2$, we obtain (1). If two functions satisfy (2), we shall call them *skew permutable*. Assume, now, that $K_1(x, y)$, $K_2(x, y)$ are symmetric functions of their arguments. We have the following lemma.

LEMMA. *The Volterra product of two symmetric functions is itself a symmetric function if and only if the two functions are skew permutable.*

Let

$$K(x, y) = \int_x^y K_1(x, \xi)K_2(\xi, y)d\xi = K_1K_2;$$

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* Presented to the Society, September 9, 1931.

† I wish to thank G. C. Evans for suggestions given me during the preparation of this paper.

‡ In this part of the paper we are concerned entirely with Volterra composition.

then

$$K(y, x) = \int_y^z K_1(y, \xi)K_2(\xi, x)d\xi,$$

and since K_1 and K_2 are symmetric functions,

$$K(y, x) = -\int_x^y K_2(x, \xi)K_1(\xi, y)d\xi = -K_2K_1.$$

Hence $K(x, y) = K(y, x)$, if and only if $K_1K_2 = -K_2K_1$, that is, if and only if the functions K_1, K_2 are skew permutable.

Hence the problem of finding two symmetric functions whose Volterra product is symmetric is equivalent, by means of the Lemma, to finding two skew permutable functions.

Before treating the general case, we show that *if a continuous function is skew permutable with unity, then the function is identically zero.**

If $K_2(x, y)$ is such a function, then

$$\int_x^y K_2(\xi, y)d\xi = -\int_x^y K_2(x, \xi)d\xi.$$

Calling the common value of the two integrals $\phi(x, y)$, we have

$$\frac{\partial\phi}{\partial x} = -K_2(x, y), \qquad \frac{\partial\phi}{\partial y} = -K_2(x, y),$$

and $\phi(x, y)$ must satisfy the equation

$$\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y} = 0,$$

that is,

$$\phi = \phi(y + x).$$

But $\phi(x, x) \equiv 0$, hence $\phi(y+x) \equiv 0$,† and therefore $K_2(x, y)$ is identically zero. We now state the following theorem.

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* This is in marked contrast to functions permutable with unity. These are functions of $(y-x)$, the so-called functions of closed cycle of Volterra.

† Consider $\phi = \phi(y+x)$. Along the lines $x+y=c$ in the XY plane, $\phi = c$. But along the line $y=x$ in this plane, $\phi \equiv 0$. Hence $\phi(y+x)$ vanishes identically.

THEOREM 1. *If the Volterra product of two continuous, symmetric functions, $K_1(x, y)$, $K_2(x, y)$, where K_1 is of the first order, with continuous first partial derivatives and with $\partial^2 K_1/\partial x \partial y$ continuous, is a symmetric function, then one of the functions is identically zero.*

Now if a function $F(x, y)$ is of the first order, we may form the function

$$F_1(x_1, y_1) = a(x_1)b(y_1)F[m(x_1), m(y_1)],$$

where we have written

$$x = m(x_1), \quad y = m(y_1), \quad m'(x_1) \neq 0,$$

and the functions a and b satisfy the relation

$$a(x_1)b(x_1) = m'(x_1).$$

It may be shown* that if

$$a = e^{-\mu}, \quad b = \frac{1}{F(x, x)} e^{\mu},$$

where

$$\mu = \int \frac{\left(\dfrac{\partial F}{\partial x}\right)_{y=x}}{F(x, x)} dx,$$

then $F_1(x_1, y_1)$ is such that

$$F_1(x_1, x_1) = 1, \quad \left(\frac{\partial F_1}{\partial x_1}\right)_{y_1=x_1} = \left(\frac{\partial F_1}{\partial y_1}\right)_{y_1=x_1} = 0.$$

We say that the function F_1 is in *canonical form*.

We shall assume that our function $K_1(x, y)$ is in canonical form. We prove our theorem by showing that if $K_2(x, y)$ is a continuous function, skew permutable with K_1, then K_2 vanishes identically.† Then an application of the Lemma yields the theorem.

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* See, for example, Volterra et Pérès, *Leçons sur la Composition*, Paris, 1924, p. 38.

† To show this it is unnecessary to assume K_1, K_2 symmetric. Their symmetry enters, however, in applying the Lemma.

Suppose that $K_2(x, y)$ is a continuous function satisfying the condition $K_1K_2 = -K_2K_1$. Calling the common value of each side ϕ, we have

$$\phi(x, y) = \int_x^y K_1(x, \xi)K_2(\xi, y)d\xi = -\int_x^y K_2(x, \xi)K_1(\xi, y)d\xi,$$

with $\phi(x, x) \equiv 0$.

Differentiating, and recalling that $K_1(x, x) = 1$, we obtain

$$\frac{\partial \phi}{\partial x} = -K_2(x, y) + \int_x^y \frac{\partial K_1(x, \xi)}{\partial x} K_2(\xi, y)d\xi,$$

$$\frac{\partial \phi}{\partial y} = -K_2(x, y) - \int_x^y K_2(x, \xi)\frac{\partial K_1}{\partial y}(\xi, y)d\xi.$$

Solving these two Volterra integral equations in $K_2(x, y)$ we get

$$K_2(x, y) = -\frac{\partial \phi}{\partial x} - \int_x^y k(x, \xi)\frac{\partial \phi(\xi, y)}{\partial \xi} d\xi,$$

$$K_2(x, y) = -\frac{\partial \phi}{\partial y} + \int_x^y \frac{\partial \phi(x, \xi)}{\partial \xi} g(\xi, y)d\xi,$$

where $k(x, y)$, $g(x, y)$ are the resolvent kernels for $\partial K_1/\partial x$ and $\partial K_1/\partial y$ respectively.

Performing an integration by parts in each of the last two equations, and taking into account that $(\partial K_1/\partial x)_{y=x} = (\partial K_1/\partial y)_{y=x} = 0$, we have

$$K_2(x, y) = -\frac{\partial \phi}{\partial x} + \int_x^y \frac{\partial k(x, \xi)}{\partial \xi} \phi(\xi, y)d\xi,$$

$$K_2(x, y) = -\frac{\partial \phi}{\partial y} - \int_x^y \phi(x, \xi)\frac{\partial g(\xi, y)}{\partial \xi} d\xi.$$

From these two equations, and the relation*

$$\frac{\partial k(x, y)}{\partial y} = \frac{\partial g(x, y)}{\partial x} = \eta(x, y),$$

* This is due to Pérès; see Volterra et Pérès, *Leçons sur la Composition*, p. 40.

we finally obtain ϕ, given by means of the integro-differential equation

$$\frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial y} = \int_x^y [\phi(x, \xi)\eta(\xi, y) + \eta(x, \xi)\phi(\xi, y)]d\xi.$$

If we denote the right-hand member by $\mathcal{E}(x, y)$, we have

$$\frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial y} = \mathcal{E}(x, y),$$

from which

$$\phi(x, y) = \int_0^x \mathcal{E}(x, y)dx + \theta(y + x), \tag{3}$$

with θ an arbitrary function. But $\phi(x, x) \equiv 0$, whence, since $\mathcal{E}(x, x) \equiv 0$, we have $\theta(2x) \equiv 0$. Then $\theta(y+x) \equiv 0$ and (3) becomes

$$\phi(x, y) = \int_0^x \mathcal{E}(x, y)dx,$$

a homogeneous Volterra integral equation for ϕ. But this equation has as its only continuous solution $\phi \equiv 0$. Then $K_2 \equiv 0$, and the theorem is proved.

COROLLARY. *The theorem is valid for functions $K_1(x, y)$ of any definite order.*

2. *Fredholm Composition.* In this section we deal exclusively with composition of the second kind.* It is well known that if $K(x, y)$ is a symmetric kernel, then all of its iterated kernels, $K_n(x, y)$, are symmetric, and since

$$K_n = \int_a^b K(x, \xi)K_{n-1}(\xi, y)d\xi,$$

we have an example of two symmetric functions whose Fredholm product is likewise symmetric. We seek now to characterize those symmetric functions whose Fredholm product is also symmetric, at least in the case of both functions having only a finite number of characteristic values.

* Volterra, *Questioni generali sulle equazioni integrali ed integro-differenziali*, Rendiconti dei Lincei, 1er sem., 1910, p. 178. See also, Volterra, *Leçons sur les Fonctions des Lignes*, Paris, 1913, p. 179.

Let $K(x, y)$, $G(x, y)$ be two continuous, symmetric kernels; then

$$KG(x, y) = \int_a^b K(x, \xi)G(\xi, y)d\xi,$$

$$KG(y, x) = \int_a^b K(y, \xi)G(\xi, x)d\xi,$$

$$= \int_a^b G(x, \xi)K(\xi, y)d\xi,$$

$$= GK(x, y).$$

Thus the Fredholm product of two symmetric kernels is symmetric if and only if $KG = GK$; that is, if and only if the kernels K, G are permutable of the second kind.

Now Volterra has shown* that the problem of permutability of the second kind can be reduced (at least in the case of kernels having a finite number of characteristic constants) to the problem of permutability of square matrices. In place of applying the complicated general theory (in which, of course, no hypothesis of symmetry on the part of the kernels is made) we shall obtain the characterization sought by methods more germane to the problem.

If, now, $K(x, y)$, $G(x, y)$ are symmetric kernels such that

$$P(x, y) = \int_a^b G(x, \xi)K(\xi, y)d\xi = P(y, x),$$

then it is evident that $\int_a^b G_r(x, \xi)K_s(\xi, y)d\xi$ is also a symmetric function. Suppose $K(x, y)$, $G(x, y)$ admit the distinct characteristic constants $\lambda_1, \lambda_2, \cdots, \lambda_k$ and $\mu_1, \mu_2, \cdots, \mu_g$ respectively. Then we may write

$$K_r(x, y) = \sum_{i=1}^{k} \frac{\phi_i(x)\phi_i(y)}{\lambda_i^r}, \quad (r = 1, 2, 3, \cdots),$$

$$G_s(x, y) = \sum_{j=1}^{g} \frac{\theta_j(x)\theta_j(y)}{\mu_j^s}, \quad (s = 1, 2, 3, \cdots),$$

* Volterra, *Leçons sur les Fonctions des Lignes*, Chapter 12.

where ϕ_i, θ_j are the characteristic functions of K and G respectively, corresponding to the characteristic values λ_i and μ_j. Now, in order that the Fredholm product KG be symmetric, it is necessary that

$$(4) \qquad \int_a^b K_r(x, \xi)G_s(\xi, y)d\xi = \int_a^b K_r(y, \xi)G_s(\xi, x)d\xi,$$

for all positive integers r, s. Writing

$$(5) \qquad \begin{aligned} \Phi_i(x, y) &= \phi_i(x)\phi_i(y), \\ \Theta_j(x, y) &= \theta_j(x)\theta_j(y), \end{aligned}$$

we have

$$(6) \qquad \int_a^b K_r(x, \xi)G_s(\xi, y)d\xi = \sum_{i=1}^{k} \sum_{j=1}^{g} \frac{\int_a^b \Phi_i(x, \xi)\Theta_j(\xi, y)d\xi}{\lambda_i^r \mu_j^s};$$

and hence, substituting in (4),

$$\sum_{i=1}^{k} \sum_{j=1}^{g} \left[\frac{\int_a^b \Phi_i(x, \xi)\Theta_j(\xi, y)d\xi - \int_a^b \Phi_i(y, \xi)\Theta_j(\xi, x)d\xi}{\lambda_i^r \mu_j^s} \right] = 0$$

for all positive integral values of r, s.

We write the above equations in the form

$$(7) \qquad \sum_{i=1}^{k} \frac{1}{\lambda_i^r} \left[\sum_{j=1}^{g} \frac{\int_a^b \Phi_i(x, \xi)\Theta_j(\xi, y)d\xi - \int_a^b \Phi_i(y, \xi)\Theta_j(\xi, x)d\xi}{\mu_j^s} \right] = \sum_{i=1}^{k} \frac{1}{\lambda_i^r} X_i = 0,$$

and agree to hold s fixed, while r takes on the values $1, 2, \cdots$. We obtain in this way an infinite number of homogeneous equations that the k functions X_i must satisfy. Consider the first k of these equations. The determinant of the coefficients is the Vandermond determinant $|1/\lambda_i^r|$, $(i, r=1, 2, \cdots, k)$, of order k. This determinant is not zero, and hence the first k equations of the set have no solution other than $X_i=0$, $(i=1, 2, \cdots, k)$.

Whence this is the only solution of the complete set of equations. Thus

$$\sum_{j=1}^{g} \frac{1}{\mu_j^s}\left[\int_a^b \Phi_i(x, \xi)\Theta_j(\xi, y)d\xi - \int_a^b \Phi_i(y, \xi)\Theta_j(\xi, x)d\xi\right] = \sum_{j=1}^{g} \frac{1}{\mu_j^s}\Gamma_j = 0.$$

Again we have an infinite set of equations in g functions Γ_j and the determinant of the coefficients appearing in the first g of the equations is the non-vanishing Vandermond determinant $|1/\mu_j^s|$, $(j, s=1, 2, \cdots, g)$. Hence the only solution of the system is $\Gamma_j=0$, and we have

$$\int_a^b \Phi_i(x, \xi)\Theta_j(\xi, y)d\xi = \int_a^b \Phi_i(y, \xi)\Theta_j(\xi, x)d\xi, \tag{8}$$

for each $i=1, 2, \cdots, k$ and $j=1, 2, \cdots, g$, as the necessary conditions that the Fredholm product KG be symmetric. These conditions are evidently sufficient, as an inspection of (6) assures us that if these conditions are satisfied, then

$$\int_a^b K_r(x, \xi)G_s(\xi, y)d\xi = \int_a^b K_r(y, \xi)G_s(\xi, x)d\xi.$$

We state the foregoing in the form of the following theorem.

THEOREM 2. *In order that the Fredholm product of two symmetric kernels, admitting only a finite number of characteristic constants, be symmetric, it is necessary and sufficient that the conditions*

$$\int_a^b \Phi_i(x, \xi)\Theta_j(\xi, y)d\xi = \int_a^b \Phi_i(y, \xi)\Theta_j(\xi, x)d\xi,$$

be satisfied for each $i=1, 2, \cdots, k; j=1, 2, \cdots, g$.

It is evident that these conditions are *highly restrictive*. Substituting from (5), conditions (8) reduce to

$$c_{ij}\phi_i(x)\theta_j(y) = c_{ij}\theta_j(x)\phi_i(y),$$

where

$$c_{ij} = \int_a^b \phi_i(\xi)\theta_j(\xi)d\xi.$$

Thus the conditions are satisfied if and only if the characteristic functions of K and G are orthogonal, or satisfy the relations

$$\frac{\phi_i(x)}{\theta_j(x)} = \frac{\phi_i(y)}{\theta_j(y)} = \text{a constant, say } k_{ij};$$

so that

$$\phi_i(x) = k_{ij}\theta_j(x).$$

That the class of functions $G(x, y)$, symmetric, and forming with a symmetric function $K(x, y)$ a symmetric Fredholm product, is so intimately related to the function $K(x, y)$ in the case of each of the functions having a finite number of characteristic constants, is a result that (it appears to the writer) could hardly have been anticipated. This is even more striking when the result is viewed as a theorem giving necessary and sufficient conditions for the permutability of the second kind of two functions satisfying the conditions of the theorem. In addition to exhibiting a fact that does not seem to have found a place in the literature of permutable functions of the second kind, the simplicity of the method used is in marked contrast to the more complicated procedure usual in treatments of this subject.

It might appear at first that using the well known development of the iterated kernels in terms of the characteristic functions

$$K_r(x, y) = \sum_{i=1}^{\infty} \frac{\phi_i(x)\phi_i(y)}{\lambda_i^r}, \qquad (r \geqq 2),$$

$$G_s(x, y) = \sum_{j=1}^{\infty} \frac{\theta_j(x)\theta_j(y)}{\mu_j^s}, \qquad (s \geqq 2),$$

both series converging absolutely and uniformly, we might treat the case for which G and K admit infinitely many characteristic values by means of an obvious extension of the previous analysis. That this is not the case is due to the fact that when we consider the infinite set of homogeneous equations in an infinity of unknowns that is the analog of the set of equations (7), the infinite Vandermond determinant

$$\left|\frac{1}{\lambda_i^r}\right|, \qquad \begin{pmatrix} r = 2, 3, \cdots \\ i = 1, 2, \cdots \end{pmatrix},$$

converges to zero. This is immediate upon applying two theorems due to T. Gazzaniga.* Indeed, these theorems are sufficient to establish in general that the infinite Vandermond determinant formed by taking the reciprocals of the infinitely many zeros of a transcendental integral function converges to zero. Thus, the foregoing method is not well adapted to kernels with an infinite number of characteristic values.

Since we have shown that the problem is equivalent to functions K, G being permutable of the second kind, the general theory may be applied to determine the conditions under which this takes place. This is a problem of much difficulty. It is possible that modifying Volterra's definition of composition so that

$$E(x, y) = K(x, y) + G(x, y) + \int_z^y K(x, \xi)G(\xi, y)d\xi,$$

is defined to be the composition of K and G (which leaves the condition of permutability unaltered) may make the problem capable of being handled by means of the algebra of functions introduced and developed by Griffith C. Evans.†

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* *Sui determinanti d'ordine infinito*, Annali di Matematica, (2), vol. 26, p. 205; *Intorno ad un tipo di determinanti d'ordine infinito*, ibid., (3), vol. 1, pp. 83–94.

† See his Cambridge Colloquium Lectures, *Functionals and their Applications*, 1918, p. 119; see also *Sopra l'algebra delle funzioni permutabili*, Atti dei Lincei, vol. 8 (1911), p. 6.

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ERRATA

Page 92, line 20: *insert T between* leaves *and* invariant.
Page 99, lines 21 and 26: *for* $R^{-9}SR$ *read* $R^{-9}SR^{9}$.
Page 135, line 15: *for* No. 37–1–68–t *read* No. 37–1–69–t.
Page 136, line 30: *for* No. 37–1–66–t *read* No. 37–1–81–t.
Page 136, line 36: *for* No. 37–1–38 *read* No. 37–1–28.
Page 138, line 4: *for* No. 37–1–89 *read* No. 36–11–429.
Page 172, line 36: *for* E. S. Wilks *read* S. S. Wilks.
Page 173, line 28: *for* H. T. Davies *read* H. T. Davis.
Page 323, line 1: *for* A. E. Craig *read* A. T. Craig.
Page 661, line 38: *for* G. Tobetti *read* G. Abetti.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

NEW ORLEANS, ANNUAL MEETING, December 28-31, 1931.

Abstracts must be in the hands of Associate Secretary Tomlinson Fort, 501 West 116th St., New York City, not later than November 28, 1931. Abstracts received by November 5 will appear in the November issue of the Bulletin. Professor E. R. Hedrick will deliver his retiring presidential address on the topic *Non-analytic functions of a complex variable.* At sessions to be held jointly with the American Physical Society, Professor P. W. Bridgman will deliver the ninth Josiah Willard Gibbs Lecture on the topic *Statistical mechanics and the second law of thermodynamics,* Professor G. A. Bliss will deliver his retiring presidential address for Section A of the A.A.A.S. on the topic *The calculus of variations of the quantum theory,* and addresses will be delivered by invitation of the program committee by Professor G. D. Birkhoff on *Stability and instability of physical systems,* and by Professor W. F. G. Swann on *The significance of the fundamental concepts of modern atomic theories.* The Cole Prize in the Theory of Numbers will be awarded.

The usual February Meeting in New York City will be omitted.

NEW YORK CITY, March 25-26, 1932.

Abstracts must be in the hands of Associate Secretary Tomlinson Fort, 501 West 116th St., New York City, not later than March 5. All such abstracts will appear in the March issue of the Bulletin. By invitation of the program committee, there will be a symposium on *Summability of series,* at which Professor Einar Hille will speak on *Summation of Fourier series,* Professor W. A. Hurwitz will speak on *General theorems on linear transformation of sequences,* and others will participate.

CHICAGO, April 8-9, 1932.

Abstracts must be in the hands of Associate Secretary Mark H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than March 18. All those received on or before March 6 will appear in the March issue of the Bulletin. By invitation of the program committee, Professor R. L. Wilder will deliver an address entitled *Point sets in three and higher dimensions and their investigation by means of unified analysis situs.*

LOS ANGELES, CALIFORNIA, SUMMER MEETING AND COLLOQUIUM, August 30-September 2, 1932.

Abstracts are due by July 26, 1932.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-459 Ahnaip St., Menasha, Wis., or 501 West 116th St., New York.

The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, October, 1930, p. 58), are payable to the Treasurer of the Society, Professor G. W. Mullins, 501 West 116th St., New York City.

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